The infinite series converges because the terms for fixed $j$ are dominated by a polynomial in $j$ divided by $2^j$. Now sum over $k$, getting

$$\sum_{j \geq 0} \binom{j + 1}{m} \frac{1}{2^j}.$$

Absorb the $j + 1$ and apply (5.57) to get the answer, $4(m + 1)$.

5.67 \(\binom{2n+2}{n+5}\) by (5.26), because

$$\binom{k+1}{2} = 3 \binom{k+1}{4}.$$

5.68 Using the fact that

$$\sum_{k \leq n/2} \binom{n}{k} = 2^{n-1} + \binom{n}{n/2} \left[ n \text{ is even} \right],$$

we get $n(2^{n-1} - \binom{n-1}{n/2}).$

5.69 Since $\binom{k+1}{2} + \binom{k-1}{2} \leq \binom{k+1}{2} + \binom{k}{2} \iff k < 1$, the minimum occurs when the $k$’s are as equal as possible. Hence, by the equipartition formula of Chapter 3, the minimum is

$$\left(\text{mod } m\right) \left(\frac{n}{m}\right) + \left(n \text{ mod } m\right) \left(\frac{n}{m}\right)$$

A similar result holds for any lower index in place of 2.

5.70 This is $F(-n, \frac{1}{2}; 1; 2)$; but it’s also $(-2)^{-n} \binom{2n}{n} F(-n, -n; \frac{1}{2}; -\frac{1}{2})$ if we replace by $-k$. Now $F(-n, -n; \frac{1}{2}; -\frac{1}{2}) = F(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; 1)$ by Gauss’s identity (5.111). (Alternatively, $F(-n, -n; \frac{1}{2}; -\frac{1}{2}) = 2^{-n} F(-n, \frac{1}{2}; \frac{1}{2}; -n)$ by the reflection law (5.101), and Kummer’s formula (5.94) relates this to (5.55).) The answer is 0 when $n$ is odd, $2^{-n} \binom{n}{n/2}$ when $n$ is even. (See [134, §1.2] for another derivation. This sum arises in the study of a simple search algorithm [164].)

5.71 (a) $S(z) = \sum_{k \geq 0} a_k z^{m+k}/(1-z)^{m+2k+1} = z^n(1-z)^{-n-1} A(z/(1-z)^2)$. (b) Here $A(z) = \sum_{k \geq 0} \binom{2n}{k} (-z)^k/(k+1) = (\sqrt{1+4z} - 1)/2z$, so we have $A(z/(1-z^2)) = 1 - z$. Thus $S_n = [z^n] (z/(1-z))^m = \binom{n-1}{m}$.

5.72 The stated quantity is $m(m-\ldots-1)n!$. Any prime divisor $p$ of $n$ divides the numerator at least $k - \nu(k)$ times and divides the denominator at most $k - \nu(k)$ times, since this is the number of