times 2 divides $k!$. A prime $p$ that does not divide $n$ must divide the product $m(m-n) \ldots (m-(k-1)!)$. It is as often as it divides $k!$, because $m(m-n) \ldots (m-(p'-1)!)$ is a multiple of $p'$ for all $r \geq 1$ and all $m$.

5.73 Plugging in $X_n = n!$ yields $\alpha = \beta = 1$: plugging in $X_n = r_i$ yields $\alpha = 1, \beta = 0$. Therefore the general solution is $X_n = \alpha r_i + \beta [n! \cdot n_i]$.

5.74 \binom{n+1}{k} \sim \binom{n-1}{k-1}, for $1 \leq k \leq n$.

5.75 The recurrence $S_k(n+1) = S_k(n) + r_k \mod 3(n)$ makes it possible to verify inductively that two of the $S$'s are equal and that $S_{\lfloor n \mod 3 \rfloor}[n]$ differs from them by $(-1)^n$. These three values split their sum $S_0(n) + S_1(n) + S_2(n) = 2^n$ as equally as possible, so there must be $2^n \mod 3$ occurrences of $\left\lceil \frac{2^n}{3} \right\rceil$ and $3 - 2^n \mod 3$ occurrences of $\left\lfloor \frac{2^n}{3} \right\rfloor$.

5.76 $Q_{n,k} = (n+1) \binom{n}{k} + \binom{n}{k+1}$.

5.77 The terms are zero unless $k_1 \leq \cdots \leq k_m$, when the product is the multinomial coefficient

$$\binom{k}{k_1, k_2 - k_1, \ldots, k_m - k_{m-1}}.$$

Therefore the sum over $k_1, \ldots, k_{m-1}$ is $m^{k_m}$, and the final sum over $k_m$ yields $(m^{n-1} - 1)/(m - 1)$.

5.78 Extend the sum to $k = 2m^2 + m - 1$; the new terms are $\binom{1}{k} + \binom{2}{k} + \cdots + \binom{n-1}{2m} = 0$. Since $m \perp (2m+1)$, the pairs $(k \mod m, k \mod (2m+1))$ are distinct. Furthermore, the numbers $(2j+1) \mod (2m+1)$ as $j$ varies from 0 to $2m$ are the numbers 0, 1, \ldots, 2m in some order. Hence the sum is

$$\sum_{0 \leq k < m} \binom{k}{j} = \sum_{0 \leq k < 2m+1} 2^k = 2^{m+1} - 1.$$

5.79 (a) The sum is $2^{2n-1}$, so the gcd must be a power of 2. If $n = 2^k q$ where $q$ is odd, $\binom{2^n}{k}$ is divisible by $2^k + 2$ and not by $2^k + 1$. Each $\binom{2^n}{k+1}$ is divisible by $2^{k+1}$ (see exercise 36), so this must be the gcd. (b) If $p^r \leq n + 1 < p^{r+1}$, we get the most radix $p$ carries by adding $k$ to $n$ when $k = p^r - 1$. The number of carries in this case is $r - e_p(n+1)$, and $r = e_p(L(n+1))$.

5.80 First prove by induction that $k! \geq (k/e)^k$.

5.81 Let $f_{t,m,n}(x)$ be the left-hand side. It is sufficient to show that we have $f_{t,m,n}[1] > 0$ and that $f_{t,m,n}(x) < 0$ for $0 \leq x \leq 1$. The value of $f_{t,m,n}[1]$ is $(-1)^{n-m-1}(i+m+\theta)$ by (5.23), and this is positive because the binomial coefficient has exactly $n-m$ negative factors. The inequality is true when $i = 0$, for the same reason. If $i > 0$, we have $f_{t,m,n}(x) = -i f_{t-1,m,n+1}(x)$, which is negative by induction.