Since \( \sum_{0 \leq k < m} (\zeta^{2i+1})^k = m(-1)^k[k=m] \), these terms sum to

\[
\sum_{k > n/m} \left( \frac{(1+1/m)mk - n - 1}{mk - n - 1} \right) (-z^m)^k = \sum_{k > n/m} \binom{m+1}{k} (n+1)\frac{z^m}{k^m} = \sum_{k > n/m} \binom{n-mk}{k} z^m.
\]

Incidentally, the functions \( \mathcal{B}_m(z) \) and \( \zeta^{2i+1} \mathcal{B}_{1+1/m}(\zeta^{2i+1}z)^{1/m} \) are the \( m+1 \) complex roots of the equation \( w^{m+1} = w^m = z^m \).

5.88 Use the facts that \( \int_0^\infty (e^{-t} - e^{-n}) \frac{dt}{t} = \ln n \) and \( (1 - e^{-t})/t \leq 1 \). (We have \( \binom{m}{k} = O(k^{-1}) \) as \( k \to \infty \), by (5.83); so this bound implies that Stirling’s series \( \sum k \delta_k \delta_k \) converges when \( x > 1 \). Hermite (155) showed that the sum is \( \ln \Gamma(1 + x) \).

5.89 Adding this to (5.19) gives \( y^{-r}(x+y)^{m+r} \) on both sides, by the binomial theorem. Differentiation gives

\[
\sum_{k > m} \binom{m+r}{k} \binom{m-k}{n} x^k y^m k^n = \sum_{k > m} \binom{-r}{k} \binom{m-k}{n} (-x)^k (x+y)^m k^n,
\]

and we can replace \( k \) by \( k + m + 1 \) and apply (5.15) to get

\[
\sum_{k > 0} \binom{m+r}{m+1+k} \binom{-n-1}{k} (-x)^{m+1+k} (x+y)^{-1-k-n} = \sum_{k > 0} \binom{-r}{m+1+k} \binom{-n-1}{k} x^{m+1+k} (x+y)^{-1-k-n}.
\]

In hypergeometric form, this reduces to

\[
\binom{1-r, n+1}{m+2, x/y} = \binom{1, x/y}{m+2, x+1+y} = (1 + x/y)^{-n-1} \binom{m+1+r, n+1+y}{m+2, x+y},
\]

which is the special case \( (a, b, c, z) = (n+1, m+1+r, m+2, -x/y) \) of the reflection law (5.101). (Thus (5.105) is related to reflection and to the formula in exercise 52.)

5.90 If \( r \) is a nonnegative integer, the sum is finite, and the derivation in the text is valid as long as none of the terms of the sum for \( 0 \leq k \leq r \) has zero in the denominator. Otherwise the sum is infinite, and the \( k \)th term \( \binom{k-r}{k} \binom{k-s}{k} \) is approximately \( k^r (r-s+1)!/(-r-1)! \) by (5.83). So we...