need $r > s + 1$ if the infinite series is going to converge. (If $r$ and $s$ are complex, the condition is $\Re r > \Re s + 1$, because $|k^2| = k^{2\Re r}$.) The sum is

$$F\left(-r, 1 \mid 1 \cdot s \right) \equiv \frac{\Gamma(-s)\Gamma(-s-1)}{\Gamma(-s-1)\Gamma(-s)} = \frac{s+1}{s+1 - r}$$

by (5.92); this is the same formula we found when $r$ and $s$ were integers.

5.91 (It’s best to use a program like MACSYMA for this.) Incidentally, when $c = (a+1)/2$, this reduces to an identity that’s equivalent to (5.110), in view of the Pfaff’s reflection law. For if $w = -z/(1 - z)$ we have $4w(1 - w) = -4z/(1 - z)^2$, and

$$F\left(\frac{1}{2}a, \frac{1}{2}a + b \mid 1 + a - b \right) = F\left(a, a+1-2b \mid -\frac{z}{1-z} \right)$$

$$= (1-z)^a F\left(a, b \mid 1+a-b \mid z \right).$$

5.92 The identities can be proved, as Clausen proved them more than 150 years ago, by showing that both sides satisfy the same differential equation.

One way to write the resulting equations between coefficients of $z^n$ is in terms of binomial coefficients:

$$\sum_k \binom{r}{k} \binom{r-s}{k} = \binom{2r}{n} \binom{2s}{n}$$

Another way is in terms of hypergeometrics:

$$F\left(\frac{1}{2}a, \frac{1}{2}a + b, 1 - a - n, 1 - b - n \mid 1 \right) = \frac{(2a)^n (a+b)^n (2b)^n}{(2a+2b)^n a^n b^n}.$$

$$F\left(\frac{1}{4}a, \frac{1}{4}a + b, a + b - n, -n \mid 1 \right)$$

$$= \frac{(1/4)^n (1/2 - a - b)^n (1/2 - a + b)^n}{(1 + a + b)^n (1/4 - a)^n (1/4 - b)^n}.$$

5.93 $\alpha^{-1} \prod_{j=1}^k (f(j) + \alpha)/f(j)$. (The special case when $f$ is a polynomial of degree 2 is equivalent to identity (5.133).)