6.21 The hinted number is a sum of fractions with odd denominators, so it has the form \( \frac{a}{b} \) with \( a \) and \( b \) odd. (Incidentally, Bertrand’s postulate implies that \( b_n \) is also divisible by at least one odd prime, whenever \( n > 2 \).)

6.22 \( \left| \frac{z}{k(k + z)} \right| \leq \frac{2|z|}{k^2} \) when \( k > \frac{2|z|}{2} \), so the sum is well defined when the denominators are not zero. If \( z = n \) we have \( \sum_{k=1}^{n} \left( \frac{1}{k(n + k)} \right) = H_m - H_m + H_n \), which approaches \( H_n \) as \( m \to \infty \). (The quantity \( H_{n-1} = \gamma \) is often called the psi function \( \psi(z) \).)

6.23 \( \frac{z}{e^z - 1} = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{1}{n!} B_n z^n/n! \).

6.24 When \( n \) is odd, \( T_n(x) \) is a polynomial in \( x^2 \), hence its coefficients are multiplied by even numbers when we form the derivative and compute \( T_{n+1}(x) \) by \( 6.95 \). (In fact we can prove more: The Bernoulli number \( B_{2n} \) always has 2 to the first power in its denominator, by exercise 54; hence \( 2^n k \| T_{2n+1} \iff 2^k \| (n + 1) \). The odd positive integers \( (n + 1)T_{2n+1}/2^2n \) are called Genocchi numbers (1, 1, 3, 17, 155, 2073, . . .), after Genocchi [117].)

6.25 \( 100n - nH_n < 100(n - 1) \) \( (n - 1)H_{n-1} \iff H_{n-1} > 99 \). (The least such \( n \) is approximately \( \approx e^{69} \), while he finishes at \( N \approx e^{100} \), about \( e \) times as long. So he is getting closer during the final 63% of his journey.)

6.26 Let \( u(k) = H_{k-1} \) and \( Av(k) = \frac{1}{k} \), so that \( u(k) = v(k) \). Then we have \( S_n = H_{n+1}^2 = \sum_{k=1}^{n} H_{k+1} = H_k^2 + H_{n+1}^2 = S_n = S_n^2 - S_n \).

6.27 Observe that when \( m > n \) we have \( \gcd(F_m, F_n) = \gcd(F_m, F_n) \) by \( 6.108 \). This yields a proof by induction.

6.28 (a) \( Q_n = \alpha(L_n + F_n)/2 + \beta F_n \). (The solution can also be written \( Q_n = \alpha F_n + \beta F_n \).) \( L_n = \phi^n + \bar{\phi}^n \).

6.29 When \( k = 0 \) the identity is \( 6.133 \). When \( k = 1 \) it is, essentially,

\[
K(x_1, \ldots, x_m) = K(x_1, \ldots, x_m) K(x_m, \ldots, x_n)
K(x_1, \ldots, x_{m-1}) K(x_{m+1}, \ldots, x_n);
\]

in Morse code terms, the second product on the right subtracts out the cases where the first product has intersecting dashes. When \( k > 1 \), an induction on \( k \) suffices, using both \( 6.127 \) and \( 6.132 \). (The identity is also true when one or more of the subscripts on \( K \) become -1, if we adopt the convention that \( K_{-1} = 0 \). When multiplication is not commutative, Euler’s identity remains valid if we write it in the form

\[
K_{m+n}(x_1, \ldots, x_{m+n}) K_k(x_{m+k}, \ldots, x_{m+1})
= K_{m+k}(x_1, \ldots, x_{m+k}) K_n(x_{m+n}, \ldots, x_{m-1})
+ (-1)^k K_{m-1}(x_1, \ldots, x_{m-1}) K_{n-k-1}(x_{m+n}, \ldots, x_{m+k+2}).
\]