6.40 If $6n - 1$ is prime, the numerator of
\[ \sum_{k=1}^{4n} \frac{(-1)^{k-1}}{k} = H_{4n} - H_{2n} \]
is divisible by $6n - 1$, because the sum is
\[ \sum_{k=2n}^{4n-1} \frac{1}{k} = \sum_{k=2n}^{3n} \left( \frac{1}{k} + \frac{1}{6n-1-k} \right) = \sum_{k=2n}^{3n-1} \frac{6n-1}{k(6n-1-k)}. \]
Similarly, if $6n + 1$ is prime, the numerator of \( \sum_{k=1}^{4n} \frac{(-1)^{k-1}}{k} = H_{4n} - H_{2n} \) is a multiple of 6n + 1. For 1987 we sum up to k = 1324.

6.41 \( S_{n+1} = \sum_{k=1}^{n} \left( \left\lfloor \frac{n+k}{2} \right\rfloor \right), \) hence we have \( S_{n+1} + S_n = \sum_{k=1}^{n} \left( \left\lfloor \frac{n+k}{2} \right\rfloor + \left\lfloor \frac{k+n}{2} \right\rfloor \right) = S_{n+2}. \) The answer is \( F_{n+2}. \)

6.42 \( F_n. \)

6.43 Set \( z = \frac{1}{10} \) in \( \sum_{n \geq 0} F_n z^n = z/(1 - z - z^2) \) to get \( \frac{10}{89}. \) The sum is a repeating decimal with period length 44:

\[ 0.11235 \ldots 95505 61797 75280 89887 64044 94382 02247 19101 12359 55+ \]

6.44 Replace \((m, k)\) by \((-m, -k)\) or \((k, -m)\) or \((-k, m)\), if necessary, so that \( m \geq k \geq 0. \) The result is clear if \( m = k. \) If \( m > k, \) we can replace \((m, k)\) by \((m - k, m)\) and use induction.

6.45 \( X_n = A(n) + B(n) + C(n) + D(n), \) where \( B(n) = F_n, \ A(n) = F_{n+1}, \ A(n) + B(n) = D(n) = 1, \) and \( B(n) = C(n) + 3D(n) = n. \)

6.46 \( \phi/2 \) and \( \phi^{-1}/2. \) Let \( u = \cos \theta \) and \( v = \cos \phi; \) then \( u = 2v^2 - 1 \) and \( v = 1 - 2\sin^2 \phi \). Hence \( u + v = 2(u + v)(v - u), \) and \( 4v^2 - 2v - 1 = 0. \) We can pursue this investigation to find the five complex fifth roots of unity:

\[ 1, \frac{\phi^{-1} \pm i\sqrt{2 + \phi}}{2}, \frac{-\phi \pm i\sqrt{3 - \phi}}{2} \]

6.47 \( 2^n\sqrt{5} \cdot F_n = (1 + \sqrt{5})^n - (1 - \sqrt{5})^n, \) and the even powers of \( \sqrt{5} \) cancel out. Now let \( p \) be an odd prime. Then \( \binom{p}{2k+1} \equiv 0 \) except when \( k = (p - 1)/2, \) and \( \binom{p+1}{2k+1} \equiv 0 \) except when \( k = 0 \) or \( k = (p - 1)/2; \) hence \( F_p \equiv 5^{(p-1)/2} \) and \( 2F_{p+1} \equiv 1 + 5^{(p-1)/2} \mod p. \) It can be shown that \( 5^{(p-1)/2} \equiv 1 \) when \( p \) has the form \( 10k \pm 1, \) and \( 5^{p+1}/2 \equiv -1 \) when \( p \) has the form \( 10k \pm 3. \) "Let \( p \) be any odd prime."

6.48 This must be true because (6.138) is a polynomial identity and we can set \( a, = 0. \)