6.40 If $6n - 1$ is prime, the numerator of
$$\sum_{k=1}^{4n} \frac{(-1)^{k-1}}{k} = H_{4n} - H_{2n}$$
is divisible by $6n - 1$, because the sum is
$$\sum_{k=2n}^{4n} \frac{1}{k} = \sum_{k=2n}^{3n} \left( \frac{1}{k} + \frac{1}{6n-1-k} \right) = \sum_{k=2n}^{3n-1} \frac{6n-1}{k(6n-1-k)}.$$  

Similarly if $6n + 1$ is prime, the numerator of $\sum_{k=1}^{4n} \frac{(-1)^{k-1}}{k} = H_{4n} - H_{2n}$ is a multiple of $6n + 1$. For 1987 we sum up to $k = 1324$.

6.41 $S_{n+1} = \sum_k \left( \frac{\lfloor (n+1+k)/2 \rfloor}{k} \right) = \sum_k \left( \frac{\lfloor (n+k)/2 \rfloor}{k} \right)$, hence we have $S_{n+1} + S_n = \sum_k \left( \frac{\lfloor (n+k)/2 \rfloor + 1}{k} \right) = S_{n+2}$. The answer is $F_{n+2}$.

6.42 $F_n$.

6.43 Set $z = \frac{1}{10}$ in $\sum_{n \geq 0} F_n z^n = z/(1 - z - z^2)$ to get $\frac{10}{89}$. The sum is a repeating decimal with period length 44:

$$0.1123595505617977528089887640449438202247191011235955\ldots$$

6.44 Replace $(m, k)$ by $(-m, -k)$ or $(k, -m)$ or $(-k, m)$, if necessary, so that $m \geq k \geq 0$. The result is clear if $m = k$. If $m > k$, we can replace $(m, k)$ by $(m - k, m)$ and use induction.

6.45 $X_n = A(n) + B(n) + C(n) - D(n) + E(n) + F(n)$, $A(n) = F_{n+1}$, $A(n) + B(n) - E(n) = 1$, and $B(n) - C(n) + D(n) = n$.

6.46 $\phi/2$ and $\phi^{-1}/2$. Let $u = \cos 72^\circ$ and $v = \cos 36^\circ$; then $u = 2v^2 - 1$ and $v = 1 - 2\sin^2 18^\circ = 1 - 2u^2$. Hence $u + v = 2(u+v)(v-u)$, and $4v^2 - 2v - 1 = 0$. We can pursue this investigation to find the five complex fifth roots of unity:

$$1, \quad \frac{\phi^{-1} \pm i\sqrt{2} + \phi}{2}, \quad \frac{\phi \pm i\sqrt{3} - \phi}{2}$$

6.47 $2^n\sqrt{5} F_n = (1 + \sqrt{5})^n + (1 - \sqrt{5})^n$, and the even powers of $\sqrt{5}$ cancel out. Now let $p$ be an odd prime. Then $(\frac{\phi}{2})^k \equiv 0$ except when $k = (p - 1)/2$, and $(\frac{\phi^{-1}}{2})^k \equiv 0$ except when $k = 0$ or $k = (p - 1)/2$; hence $F_p \equiv 5^{(p-1)/2}$ and $2F_{p+1} \equiv 1 + 5^{(p-1)/2}$ (mod $p$). It can be shown that $5^{(p-1)/2} \equiv 1$ when $p$ has the form $10k \pm 1$, and $5^{(p+1)/2} \equiv -1$ when $p$ has the form $10k \pm 3$.

6.48 This must be true because (6.138) is a polynomial identity and we can set $a_i = 0$.

"Let $p$ be any old prime." (See [140], p. 419.)