With regard to periodic solutions, Theorems 9.7.1 and 9.7.2 provide only partial information. From Theorem 9.7.1 we conclude that if there are closed trajectories, they must enclose the origin. Next, we calculate \( F_x(x, y) + G_y(x, y) \), with the result that
\[
F_x(x, y) + G_y(x, y) = \mu(1 - x^2). \tag{20}
\]
Then, it follows from Theorem 9.7.2 that closed trajectories, if there are any, are not contained in the strip \(|x| < 1\) where \( F_x + G_y > 0 \).

The application of the Poincaré–Bendixson theorem to this problem is not nearly as simple as for the preceding example. If we introduce polar coordinates, we find that the equation for the radial variable \( r \) is
\[
r' = \mu(1 - r^2 \cos^2 \theta) r \sin^2 \theta. \tag{21}
\]
Again, consider an annular region \( R \) given by \( r_1 \leq r \leq r_2 \), where \( r_1 \) is small and \( r_2 \) is large. When \( r = r_1 \), the linear term on the right side of Eq. (21) dominates, and \( r' > 0 \) except on the \( x \)-axis, where \( \sin \theta = 0 \) and consequently \( r' = 0 \) also. Thus, trajectories are entering \( R \) at every point on the circle \( r = r_1 \) except possibly for those on the \( x \)-axis, where the trajectories are tangent to the circle. When \( r = r_2 \), the cubic term on the right side of Eq. (21) is the dominant one. Thus \( r' < 0 \) except for points on the \( x \)-axis where \( r' = 0 \) and for points near the \( y \)-axis where \( r^2 \cos^2 \theta < 1 \) and the linear term makes \( r' > 0 \). Thus, no matter how large a circle is chosen, there will be points on it (namely, the points on or near the \( y \)-axis) where trajectories are leaving \( R \). Therefore, the Poincaré–Bendixson theorem is not applicable unless we consider more complicated regions.

It is possible to show, by a more intricate analysis, that the van der Pol equation does have a unique limit cycle. However, we will not follow this line of argument further, but turn instead to a different approach in which we plot numerically computed solutions. Experimental observations indicate that the van der Pol equation has a stable periodic solution whose period and amplitude depend on the parameter \( \mu \). By looking at graphs of trajectories in the phase plane and of \( u \) versus \( t \) we can gain some understanding of this periodic behavior.

Figure 9.7.2 shows two trajectories of the van der Pol equation in the phase plane for \( \mu = 0.2 \). The trajectory starting near the origin spirals outward in the clockwise

![FIGURE 9.7.2 Trajectories of the van der Pol equation (17) for \( \mu = 0.2 \).](image-url)