solutions to $p \alpha_n$ if and only if there are no solutions to $p^2 H_{p-1} + H_r \equiv 0 \pmod{p}$ for $0 \leq r < p$. The latter condition holds not only for $p = 5$ but also for $p = 13, 17, 23, 41,$ and $67$— perhaps for infinitely many primes. The numerator of $H_r$, is divisible by $3$ only when $n = 2, 7,$ and $22$; it is divisible by $7$ only when $n = 6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735,$ and $102728$.

6.53 Summation by parts yields

$$\frac{n+1}{(n+2)^2} \left( \frac{(-1)^m}{m+1} \left( \binom{n+2}{m+1} H_{m+1} - 1 \right) \right).$$

6.54 (a) If $m \geq p$ we have $S_m(p) \equiv S_{m-p+1}(p) \pmod{p}$, since $k^p \equiv 1$ when $1 \leq k < p$. Also $S_{p-1}(p) \equiv p-1 \equiv -1$. If $0 < m < p-1$, we can write

$$S_m(p) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} \sum_{k=0}^{p-1} k^j = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} \frac{p^j + 1}{j+1} \equiv 0$$

(b) The condition in the hint implies that the denominator of $I_{2n}$ is not divisible by any prime $p$; hence $I_{2n}$ must be an integer. To prove the hint, we may assume that $n > 1$. Then

$$B_{2n} + \frac{[p-1][2n]}{p} + \sum_{k=0}^{2n-2} \binom{2n+1}{k} B_k \frac{p^{2n-k} - k}{2n+1}$$

is an integer, by (6.78), (6.84), and part (a). So we want to verify that none of the fractions $\binom{2n+1}{k} B_k p^{2n-k} / (2n+1) = \binom{2n}{k} B_k \frac{p^{2n-k} - k + 1}{2n+1}$ has a denominator divisible by $p$. The denominator of $\binom{2n}{k} B_k p$ isn’t divisible by $p$, since $B_k$ has no $p^2$ in its denominator (by induction); and the denominator of $p^{2n-k} / (2n - k + 1)$ isn’t divisible by $p$, since $2n - k + 1 < p^{2n-k}$ when $k \leq 2n-2$; QED. (The numbers $I_{2n}$ are tabulated in [185]. Hermite calculated them through $I_{18}$ in 1875 [153]. It turns out that $I_2 = I_4 = I_6 = 1s = I_{10} = I_{12} = 1$; hence there is actually a “simple” pattern to the Bernoulli numbers displayed in the text, including $\frac{49}{2730}$ (!). But the numbers $I_{2n}$ don’t seem to have any memorable features when $n > 6$. For example, $B_{24} = -86579 \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3}$, and 86579 is prime.)

(c) The numbers $2 - 1$ and $3 - 1$ always divide $2n$. If $n$ is prime, the only divisors of $2n$ are $1, 2, n,$ and $2n$, so the denominator of $B_{2n}$ for prime $n > 2$ will be $6$ unless $2n+1$ is also prime. In the latter case we can try $4n+3, 8n+7, \ldots,$ until we eventually hit a nonprime (since $n$ divides $2''^n (1+2''^n - 1)$).

(Perhaps many primes of the form $6k + 1$.) The denominator of $B_{2n}$ can be $6$ also when $n$ has nonprime values, such as $49$. (Attention: computer programmers: Here’s an interesting condition to test, for as many primes as you can.)