6.55 The stated sum is \( \frac{m+1}{x+m+1} \binom{x+n}{n} \binom{n}{m+1} \), by Vandermonde's convolution. To get (6.70), differentiate and set \( x = 0 \).

6.56 First replace \( k^{n+1} \) by \( ((k - m) + m) n + 1 \) and expand in powers of \( k - m \); simplifications occur as in the derivation of (6.72). If \( m \geq n \) or \( m < 0 \), the answer is \((-1)^{n!} m^n / \binom{n}{m} \). Otherwise we need to take the limit of \((-1)^{n!} + (-1)^{m+1} \binom{n}{m} m^n(n + 1 + mH_n - mH_m) \).

6.57 First prove by induction that the \( n \)th row contains at most three distinct values \( A_n, B_n, C_n \); if \( n \) is even they occur in the cyclic order \([C_n, B_n, A_n, B_n, C_n]\), while if \( n \) is odd they occur in the cyclic order \([C_n, B_n, A_n, A_n, B_n]\). Also
\[
\begin{align*}
A_{2n+1} &= A_{2n} + B_{2n}, \\
B_{2n+1} &= B_{2n} + C_{2n}, \\
C_{2n+1} &= 2C_{2n}.
\end{align*}
\]

It follows that \( Q_n = A_n - C_n = F_{n+1} \). (See exercise 5.75 for wraparound binomial coefficients of order 3.)

6.58 (a) \( \sum_{n \geq 0} \frac{F_n z^n}{n!} = z[(1 - z)/(1 + z)(1 - 3z + z^2)] = \frac{1}{2}((1 - 3z)/(1 - z + 2z^2) - 2/(1 + z)) \).
(b) \( \sum_{n \geq 0} \frac{F_n^3 z^n}{n!} = z((1 - 2z - z^2)/(1 - 4z - z^2)(1 + z - z^2)) = \frac{1}{3}(2z/[1 - 4z - z^2] + 3z/[1 + z - z^2]) \).

These formulas are obtained by squaring or cubing Binet's formula (6.123) and summing on \( n \), then combining terms so that \( \phi \) and \( \bar{\phi} \) disappear. It follows that \( F_{n+1}^3 = 4F_n^3 - F_{n-1}^3 = 3(-1)^n F_n \).

(The corresponding recurrence for \( m \)th powers has been found by Jarden and Motakin [163].)

6.59 Let \( m \) be fixed. We can prove by induction on \( n \) that it is, in fact, possible to find such an \( x \) with the additional condition \( x \equiv 2 \pmod{4} \). If \( x \) is such a solution, we can move up to a solution modulo \( 3^n + 1 \) because
\[
F_{8 \cdot 3^n - 1} = 3^n.
\]

either \( x \) or \( x + 8 \cdot 3^n \) or \( x + 16 \cdot 3^n \) will do the job.

6.60 \( F_1 + 1, F_2 + 1, F_3 + 1, F_4 + 1, F_6 - 1 \) are the only cases. Otherwise the Lucas numbers of exercise 28 arise in the factorizations
\[
\begin{align*}
F_{2m} + (-1)^m &= L_{m+1} F_m + 1, \\
F_{2m+1} + (-1)^m &= L_m F_{m+1}; \\
F_{2m} - (-1)^m &= L_m F_{m+1}, \\
F_{2m+1} - (-1)^m &= L_m F_m.
\end{align*}
\]

(We have \( F_{m+1} - (-1)^n F_{n+1} = L_m F_n \) in general.)

6.61 \( 1/F_{2m} = F_m \), \( F_m = F_{2m} - F_{2m-1} / F_{2m} \) when \( m \) is even and positive. The second sum is \( 5/4 \) \( F_{3 \cdot 2^n - 1} / F_{3 \cdot 2^n} \), for \( n \geq 1 \).