in the opposite direction. The waveform of the limit cycle, as shown in Figure 9.7.7, is quite different from a sine wave.

These graphs clearly show that, in the absence of external excitation, the van der Pol oscillator has a certain characteristic mode of vibration for each value of $\mu$. The graphs of $u$ versus $t$ show that the amplitude of this oscillation changes very little with $\mu$, but the period increases as $\mu$ increases. At the same time, the waveform changes from one that is very nearly sinusoidal to one that is much less smooth.

The presence of a single periodic motion that attracts all (nearby) solutions, that is, a stable limit cycle, is one of the characteristic phenomena associated with nonlinear differential equations.

**PROBLEMS**

In each of Problems 1 through 6 an autonomous system is expressed in polar coordinates. Determine all periodic solutions, all limit cycles, and determine their stability characteristics.

1. \[ \frac{dr}{dt} = r^2(1 - r^2), \quad \frac{d\theta}{dt} = 1 \]
2. \[ \frac{dr}{dt} = r(1 - r)^2, \quad \frac{d\theta}{dt} = -1 \]
3. \[ \frac{dr}{dt} = r(r + 1)(r - 3), \quad \frac{d\theta}{dt} = 1 \]
4. \[ \frac{dr}{dt} = r(1 - r)(r - 2), \quad \frac{d\theta}{dt} = -1 \]
5. \[ \frac{dr}{dt} = \sin \pi r, \quad \frac{d\theta}{dt} = 1 \]
6. \[ \frac{dr}{dt} = r(r - 2)(r - 3), \quad \frac{d\theta}{dt} = -1 \]
7. If \( x = r \cos \theta, y = r \sin \theta \), show that \( y(dx/dt) - x(dy/dt) = -r^2(d\theta/dt) \).
8. (a) Show that the system

\[
\begin{align*}
\frac{dx}{dt} &= -y + xf(r)/r, \\
\frac{dy}{dt} &= x + yf(r)/r
\end{align*}
\]

has periodic solutions corresponding to the zeros of \( f(r) \). What is the direction of motion on the closed trajectories in the phase plane?

(b) Let \( f(r) = r(r - 2)^2(r^2 - 4r + 3) \). Determine all periodic solutions and determine their stability characteristics.

9. Determine the periodic solutions, if any, of the system

\[
\begin{align*}
\frac{dx}{dt} &= y + \frac{x}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2), \\
\frac{dy}{dt} &= -x + \frac{y}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2).
\end{align*}
\]