10. Using Theorem 9.7.2, show that the linear autonomous system

\[ \frac{dx}{dt} = a_{11}x + a_{12}y, \quad \frac{dy}{dt} = a_{21}x + a_{22}y \]

does not have a periodic solution (other than \( x = 0, y = 0 \)) if \( a_{11} + a_{22} \neq 0 \).

In each of Problems 11 and 12 show that the given system has no periodic solutions other than constant solutions.

11. \( \frac{dx}{dt} = x + y + x^3 - y^2 \), \( \frac{dy}{dt} = -x + 2y + x^2y + y^3/3 \)

12. \( \frac{dx}{dt} = -2x - 3y - xy^2 \), \( \frac{dy}{dt} = y + x^3 - x^2y \)

13. Prove Theorem 9.7.2 by completing the following argument. According to Green’s theorem in the plane, if \( C \) is a sufficiently smooth simple closed curve, and if \( F \) and \( G \) are continuous and have continuous first partial derivatives, then

\[ \int_C [F(x, y) \, dy - G(x, y) \, dx] = \int_R [F_x(x, y) + G_y(x, y)] \, dA, \]

where \( C \) is traversed counterclockwise and \( R \) is the region enclosed by \( C \). Assume that \( x = \phi(t), y = \psi(t) \) is a solution of the system (15) that is periodic with period \( T \). Let \( C \) be the closed curve given by \( x = \phi(t), y = \psi(t) \) for \( 0 \leq t \leq T \). Show that for this curve the line integral is zero. Then show that the conclusion of Theorem 9.7.2 must follow.

14. (a) By examining the graphs of \( u \) versus \( t \) in Figures 9.7.3, 9.7.5, and 9.7.7 estimate the period \( T \) of the van der Pol oscillator in these cases.

(b) Calculate and plot the graphs of solutions of the van der Pol equation for other values of the parameter \( \mu \). Estimate the period \( T \) in these cases also.

(c) Plot the estimated values of \( T \) versus \( \mu \). Describe how \( T \) depends on \( \mu \).

15. The equation

\[ u'' - \mu(1 - \frac{1}{3}u^2)u' + u = 0 \]

is often called the Rayleigh equation.

(a) Write the Rayleigh equation as a system of two first order equations.

(b) Show that the origin is the only critical point of this system. Determine its type and whether it is stable or unstable.

(c) Let \( \mu = 1 \). Choose initial conditions and compute the corresponding solution of the system on an interval such as \( 0 \leq t \leq 20 \) or longer. Plot \( u \) versus \( t \) and also plot the trajectory in the phase plane. Observe that the trajectory approaches a closed curve (limit cycle). Estimate the amplitude \( A \) and the period \( T \) of the limit cycle.

(d) Repeat part (c) for other values of \( \mu \), such as \( \mu = 0.2, 0.5, 2, \) and \( 5 \). In each case estimate the amplitude \( A \) and the period \( T \).

(e) Describe how the limit cycle changes as \( \mu \) increases. For example, make a table of values and/or plot \( A \) and \( T \) as functions of \( \mu \).

16. The system

\[ x' = 3(x + y - \frac{1}{4}x^3 - k), \quad y' = -\frac{1}{2}(x + 0.8y - 0.7) \]

is a special case of the FitzHugh–Nagumo equations, which model the transmission of neural impulses along an axon. The parameter \( k \) is the external stimulus.

(a) For \( k = 0 \) show that there is one critical point. Find this point and show that it is an asymptotically stable spiral point. Repeat the analysis for \( k = 0.5 \) and show that the critical point is now an unstable spiral point. Draw a phase portrait for the system in each case.

(b) Find the value \( k_0 \) where the critical point changes from asymptotically stable to unstable. Draw a phase portrait for the system for \( k = k_0 \).

(c) For \( k \geq k_0 \) the system exhibits an asymptotically stable limit cycle. Plot \( x \) versus \( t \) for \( k = k_0 \) for several periods and estimate the value of the period \( T \).

(d) The limit cycle actually exists for a small range of \( k \) below \( k_0 \). Let \( k_1 \) be the smallest value of \( k \) for which there is a limit cycle. Find \( k_1 \).