6.68 \( 1/k - 1/(k + z) = z/k^2 = z^2/k^3 + \), and everything converges when \(|z| < 1\).

6.69 Note that \( \prod_{k=1}^{n}(1 + z/k)e^{-z/k} = \left(\frac{n+z}{n}\right)^{n-1}e^{(\ln n + z)}z. \) If \( f(z) = \frac{d}{dz}(z!) \) we find \( f(z)/z! = H_z. \)

6.70 For \( \tan z \), we can use \( \tan z = \cot z + 2 \cot 2 \) (which is equivalent to the identity of exercise 23). Also \( z\sin z = z\cot z + \frac{1}{2}z \) has the power series \( \sum_{n \geq 0}(-1)^{n-1}(4^n - 2)B_{2n}z^{2n}/(2n)! \); and

\[
\ln \frac{\tan z}{z} = \ln \frac{\sin z}{z} \cos z = \\
\frac{\sum_{n \geq 1}(-1)^n 4^nB_{2n}z^{2n}}{(2n)(2n)!} - \frac{\sum_{n \geq 1}(-1)^n 4^n(4^n-1)B_{2n}z^{2n}}{(2n)(2n)!} = \\
\frac{\sum_{n \geq 1}(-1)^n 4^n(4^n-2)B_{2n}z^{2n}}{(2n)(2n)!},
\]

because \( \frac{d}{dz}\ln \sin z = \cot z \) and \( \frac{d}{dz}\ln \cos z = -\tan z. \)

6.71 Since \( \tan 2z = \sec 2z = (\sin z + \cos z)/(\cos z - \sin z) \), setting \( x = 1 \) in (6.94) gives \( T_n(1) = 2^nT_n \) when \( n \) is odd, \( T_n(1) = 2^nE_n \) when \( n \) is even, where \( 1/\cos z = \sum_{n \geq 0}E_{2n}z^{2n}/(2n)! \). (The \( E_n \) are called Euler numbers, not to be confused with the Eulerian numbers \( \left\langle \binom{n}{k} \right\rangle \).)

6.72 \( 2^{n+1}(2^{n+1} - 1)B_{n+1}/(n + 1), \) if \( n > 0 \). (See (7.56) and (6.92); the desired numbers are essentially the coefficients of \( \frac{1}{1-\tanh z} \).)

6.73 \( \cot(z + \pi) = \cot z \) and \( \cot(z + \frac{1}{2}\pi) = -\tan z; \) hence the identity is equivalent to

\[
\cot z = \frac{1}{2n} \sum_{k=0}^{2n-1} \cot \frac{z + k\pi}{2n},
\]

which follows by induction from the case \( n = 1 \). The stated limit follows since \( z\cot z \to 1 \) as \( z \to 0. \) It can be shown that term-by-term passage to the limit is justified, hence (6.88) is valid. (Incidentally, the general formula

\[
\cot z = \frac{1}{n} \sum_{k=0}^{n-1} \cot \frac{z + k\pi}{n}
\]

is also true. It can be proved from (6.88), or from

\[
\frac{1}{e^{\pi i}} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{e^{\pi i(2k+1)/n}},
\]

which is equivalent to the partial fraction expansion of \( 1/(z^n - 1). \))