and we have $\sum_{k=0}^{t} a_k \phi^k = 0$ by taking the limit as $n \to \infty$. Hence $Q(x, y)$ is a multiple of $U(x, y)$, say $A(x, y)U(x, y)$. But $U(F_{n+1}, F_n) = (-1)^n$ and $n$ is even, so $P_0(x, y) = P(x, y) - (U(x, y) - 1)A(x, y)$ is another polynomial such that $P_0(F_{n+1}, F_n) \equiv 0$. The total degree of $P_0$ is less than $t$, so $P_0$ is a multiple of $U \sim 1$ by induction on $t$.

Similarly, $P(x, y)$ is divisible by $U(x, y) + 1$ if $P(F_{n+1}, F_n) = 0$ for infinitely many odd values of $n$. A combination of these two facts gives the desired necessary and sufficient condition: $P(x, y)$ is divisible by $U(x, y)^t - 1$.

6.78 First add the digits without carrying, getting digits 0, 1, and 2. Then use the carry rules

$$0(d+1)(e+1) \rightarrow 1de,$$
$$0(d+2)(e) \rightarrow 1d0(e+1),$$

always applying the leftmost applicable carry. This process terminates because the binary value obtained by reading $(b, \ldots, b_2)_F$ as $(b, \ldots, b_2)_2$ increases whenever a carry is performed. But a carry might propagate to the right of the “Fibonacci point”; for example, $(1, \ldots, 1)_F$ becomes $(10,01)_F$. Such rightward propagation extends at most two positions; and those two digit positions can be zeroed again by using the text’s “add 1” algorithm if necessary.

Incidentally, there’s a corresponding “multiplication” operation on nonnegative integers: If $m = F_{k_1} + \cdots + F_{k_r}$ and $n = F_{k_1} + \cdots + F_{k_s}$ in the Fibonacci number system, let $m \circ n = \sum_{i=1}^{r} \sum_{j=1}^{s} F_{i+j+k_r}$, by analogy with multiplication of binary numbers. (This definition implies that $m \circ n \approx \sqrt{5}mn$ when $m$ and $n$ are large, although $1 \circ n \approx \phi n$.) Fibonacci addition leads to a proof of the associative law $1 \circ (m \circ n) = (1 \circ m) \circ n$.

6.79 Yes; for example, we can take

$$A_0 = 33163563598274737472200656430763;$$
$$A_1 = 1510028911088401971189590305498785 .$$

The resulting sequence has the property that $A_n$ is divisible by (but unequal to) $p_k$ when $n \mod m_k = r_k$, where the numbers $(p_k, m_k, r_k)$ have the following 18 respective values:

$$\begin{align*}
(3, 4, 1) & \quad (2, 3, 2) & \quad (5, 5, 1) \\
(7, 8, 3) & \quad (17, 9, 4) & \quad (11, 10, 2) \\
(47, 16, 7) & \quad (19, 18, 10) & \quad (61, 15, 3) \\
(2207, 32, 15) & \quad (53, 27, 16) & \quad (31, 30, 24) \\
(1087, 64, 31) & \quad (109, 27, 7) & \quad (41, 20, 10) \\
(4481, 64, 63) & \quad (5779, 54, 52) & \quad (2521, 60, 60)
\end{align*}$$