Therefore
\[ \lambda_1 = -\frac{8}{3}, \quad \lambda_2 = -11 - \sqrt{81 + 40r}, \quad \lambda_3 = -11 + \sqrt{81 + 40r}. \] (7)

Note that all three eigenvalues are negative for \( r < 1 \); for example, when \( r = 1/2 \), the eigenvalues are \( \lambda_1 = -8/3, \lambda_2 = -10.52494, \lambda_3 = -0.47506 \). Hence the origin is asymptotically stable for this range of \( r \) both for the linear approximation (5) and for the original system (1). However, \( \lambda_3 \) changes sign when \( r = 1 \) and is positive for \( r > 1 \). The value \( r = 1 \) corresponds to the initiation of convective flow in the physical problem described earlier. The origin is unstable for \( r > 1 \); all solutions starting near the origin tend to grow except for those lying precisely in the plane determined by the eigenvectors associated with \( \lambda_1 \) and \( \lambda_2 \) [or, for the nonlinear system (1), in a certain surface tangent to this plane at the origin].

Next let us consider the neighborhood of the critical point \( P_2(\sqrt{8(r-1)/3}, \sqrt{8(r-1)/3}, r-1) \) for \( r > 1 \). If \( u, v, \) and \( w \) are the perturbations from the critical point in the \( x, y, \) and \( z \) directions, respectively, then the approximating linear system is
\[
\begin{pmatrix} u \\ v \\ w \end{pmatrix}' = \begin{pmatrix} -10 & 10 & 0 \\ \sqrt{8(r-1)/3} & -1 & -\sqrt{8(r-1)/3} \\ \sqrt{8(r-1)/3} & \sqrt{8(r-1)/3} & -8/3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}.
\] (8)

The eigenvalues of the coefficient matrix of Eq. (8) are determined from the equation
\[ 3\lambda^3 + 41\lambda^2 + 8(r+10)\lambda + 160(r-1) = 0, \] (9)
which is obtained by straightforward algebraic steps that are omitted here. The solutions of Eq. (9) depend on \( r \) in the following way:

For \( 1 < r < r_1 \cong 1.3456 \) there are three negative real eigenvalues.
For \( r_1 < r < r_2 \cong 24.737 \) there are one negative real eigenvalue and two complex eigenvalues with negative real part.
For \( r_2 < r \) there are one negative real eigenvalue and two complex eigenvalues with positive real part.

The same results are obtained for the critical point \( P_3 \). Thus there are several different situations.

For \( 0 < r < 1 \) the only critical point is \( P_1 \) and it is asymptotically stable. All solutions approach this point (the origin) as \( t \to \infty \).
For \( 1 < r < r_1 \) the critical points \( P_2 \) and \( P_3 \) are asymptotically stable and \( P_1 \) is unstable. All nearby solutions approach one or the other of the points \( P_2 \) and \( P_3 \) exponentially.
For \( r_1 < r < r_2 \) the critical points \( P_2 \) and \( P_3 \) are asymptotically stable and \( P_1 \) is unstable. All nearby solutions approach one or the other of the points \( P_2 \) and \( P_3 \); most of them spiral inward to the critical point.
For \( r_2 < r \) all three critical points are unstable. Most solutions near \( P_2 \) or \( P_3 \) spiral away from the critical point.

However, this is by no means the end of the story. Let us consider solutions for \( r \) somewhat greater than \( r_2 \). In this case \( P_1 \) has one positive eigenvalue and each of \( P_2 \) and \( P_3 \) has a pair of complex eigenvalues with positive real part. A trajectory can approach any one of the critical points only on certain highly restricted paths. The