6.85 The property holds if and only if \( N \) has one of the seven forms \( 5^k, 2 \cdot 5^k, 4 \cdot 5^k, 3 \cdot 5^k, 6 \cdot 5^k, 7 \cdot 5^k, 14 \cdot 5^k \).

6.86 A candidate for the case \( n \mod 1 = \frac{1}{2} \) appears in [179, section 6], although it may be best to multiply the integers discussed there by some constant involving \( \sqrt{\pi} \).

6.87 (a) If there are only finitely many solutions, it is natural to conjecture that the same holds for all primes. (b) The behavior of \( b_n \) is quite strange: We have \( b_n = \text{lcm}(1, \ldots, n) \) for \( 968 \leq n \leq 1066 \); on the other hand, \( b_{600} = \frac{\text{lcm}(1, \ldots, 600)}{3^3 \cdot 5^2 \cdot 43} \). Andrew Odlyzko observes that \( p \) divides \( \text{lcm}(1, \ldots, n)/b_n \) if and only if \( kp^m \leq n < (k+1)p^m \) for some \( m \geq 1 \) and some \( k < p \) such that \( p \) divides the numerator of \( H_k \). Therefore infinitely many such \( n \) exist if it can be shown, for example, that almost all primes have only one such value of \( k \) (namely \( k = p - 1 \)).

6.88 (Brent [33] found the surprisingly large partial quotient 1568705 in \( e^\gamma \), but this seems to be just a coincidence. For example, Gosper has found even larger partial quotients in \( \pi \): The 453,294th is 12996958 and the 411,504,931st is 878783625.)

6.89 Consider the generating function \( \sum_{m,n \geq 0} \frac{m+n}{m} w^m z^n \), which has the form \( \sum_n (w F(a, b, c) + z F(a', b', c'))^n \), where \( F(a, b, c) \) is the differential operator \( a + b z^4 + c z^5 \).

7.1 Substitute \( z^4 \) for \( w \) and \( z \) for \( z \) in the generating function, getting \( 1/(1 - z^4 - z^2) \). This is like the generating function for \( T \), but with \( z \) replaced by \( z^2 \). Therefore the answer is zero if \( m \) is odd, otherwise \( F_{m/2+1} \).

7.2 \( G(z) = 1/(1 - 2z) + 1/(1 - 3z); \hat{G}(z) = e^{2z} + e^{3z} \).

7.3 Set \( z = 1/10 \) in the generating function, getting \( \frac{10}{9} \ln \frac{10}{9} \).

7.4 Divide \( P(z) \) by \( Q(z) \), getting a quotient \( T(z) \) and a remainder \( P_0(z) \) whose degree is less than the degree of \( Q \). The coefficients of \( T(z) \) must be added to the coefficients \( [z^n] P_0(z)/Q(z) \) for small \( n \). (This is the polynomial \( T(z) \) in (7.28).)

7.5 This is the convolution of \( (1 + z^2)^r \) with \( (1 + z)^r \), so
\[
S(z) = (1 + z + z^2 + z^3)^r.
\]
Incidentally, no simple form is known for the coefficients of this generating function; hence the stated sum probably has no simple closed form. (We can use generating functions to obtain negative results as well as positive ones.)