Hence $g_{2n+1} = 0$ and $g_{2n} = (-1)^n (2n)! C_{n-1}$, for all $n > 0$.

7.15 There are $\binom{n+k}{k}$ partitions with $k$ other objects in the subset containing $n+1$. Hence $\hat{B}'(z) = e^{z} B(z)$. The solution to this differential equation is $\hat{B}(z) = e^{z+c}$, and $c = -1$ since $B(0) = 1$. (We can also get this result by summing (7.49) on $m$, since $b_n = \sum m \{n\}$.)

7.16 One way is to take the logarithm of

$$B(z) = 1/(1-z)^{a_1}(1-z^2)^{a_2}(1-z^3)^{a_3}(1-z^4)^{a_4} \cdots,$$

then use the formula for $\ln \frac{1}{1-z}$ and interchange the order of summation.

7.17 This follows since $\int_0^\infty t^n e^{-t} dt = n!$. There's also a formula that goes in the other direction:

$$\hat{G}(z) = \frac{1}{2\pi} \int_0^{2\pi} G(ze^{-i\theta}) e^{i\theta} d\theta.$$

7.18 (a) $\zeta(z - \frac{1}{2})$; (b) $-L'(z)$; (c) $\zeta(z)/\zeta(2z)$. Every positive integer is uniquely representable as $m^2 q_i$, where $q_i$ is squarefree.

7.19 If $n > 0$, the coefficient $[z^n] \exp(x \ln F(z))$ is a polynomial of degree $n$ in $x$ that's a multiple of $x$. The first convolution formula comes from equating coefficients of $z^n$ in $F(z)^y F(z)^{y-1}$. The second comes from equating coefficients of $z^{n-1}$ in $F'(z) F(z)^y F(z)^{y-1}$, because we have

$$F'(z) F(z)^{y-1} = x^{-1} \frac{\partial}{\partial z} (F(z)^{y}) = x^{-1} \sum_{n \geq 0} n f_n(x) z^{n-1}.$$

(Further convolutions follow by taking $\partial/\partial x$, as in (7.43).)

7.20 Let $G(z) = \sum n \geq 0 g_n z^n$. Then

$$z^l G^{(k)}(z) = \sum_{n \geq 0} n^k g_n z^{n-k-l} = \sum_{n \geq 0} (n + k + 1)^l g_{n+k+l} z^n$$

for all $k, l \geq 0$, if we regard $g_n = 0$ for $n < 0$. Hence if $p_0(z), \ldots, p_d(z)$ are polynomials, not all zero, having maximum degree $d$, then there are polynomials $p_0(n), \ldots, p_{m+d}(n)$ such that

$$p_0(z) G(z) + \cdots + p_d(z) G^{[m]}(z) = \sum_{n \geq 0} \sum_{j=0}^{m+d} p_j(n) g_{n+j+d} z^n.$$

Therefore a differentially finite $G(z)$ implies that

$$\sum_{j=0}^{m+d} p_j(n+d) g_{n+j} = 0,$$

for all $n \geq 0$. 