The converse is similar. (One consequence is that \( G(z) \) is differentiably finite if and only if the corresponding egf, \( \hat{G}(z) \), is differentiably finite.)

7.21 This is the problem of giving change with denominations 10 and 20, so 
\[ G(z) = \frac{1}{1 - (1 - z^{10}) (1 - z^{20})} = \hat{G}(z^{10}) \]
where \( \hat{G}(z) = 1 - (1 - z)^{-1} + \frac{1}{2} (1 + z)^{-1} \), so 
\[ [z^n] \hat{G}(z) = \frac{1}{4} (2n + 3 + (-1)^n) \]. Setting \( n = 50 \) yields 26 ways to make the payment. (b) \( \hat{G}(z) = \frac{[1 + z]}{[1 - z^2]}^2 = \frac{1}{(1 + 2z^2 + 3z^4 + \cdots)} \), so 
\[ [z^n] \hat{G}(z) = \frac{n}{2} + 1 \]. (Compare this with the value \( N_n = \left\lfloor \frac{n}{5} \right\rfloor + 1 \) in the text's coin-changing problem. The bank robber's problem is equivalent to the problem of making change with pennies and tuppences.)

7.22 Each polygon has a "base" (the line segment at the bottom). If \( A \) and \( B \) are triangulated polygons, let \( A \triangle B \) be the result of pasting the base of \( A \) to the upper left diagonal of \( \triangle \), and pasting the base of \( B \) to the upper right diagonal. Thus, for example,

![Diagram of a triangulated polygon](Image)

(The polygons might need to be warped a bit and/or banged into shape.) Every triangulation arises in this way, because the base line is part of a unique triangle and there are triangulated polygons \( A \) and \( B \) at its left and right.

Replacing each triangle by \( z \) gives a power series in which the coefficient of \( z^n \) is the number of triangulations with \( n \) triangles, namely the number of ways to decompose an \( (n+2) \)-gon into triangles. Since \( P = 1 + zP^2 \), this is the generating function for Catalan numbers \( C_0 + C_1 z + C_2 z^2 + \ldots \); the number of ways to triangulate an \( n \)-gon is \( C_{n-2} = \frac{(2n-4)!}{(n-2)!} \).

7.23 Let \( a_n \), be the stated number, and \( b_n \), the number of ways with a 2x1 x 1 notch missing at the top. By considering the possible patterns visible on the top surface, we have

\[
\begin{align*}
  a_n &= 2a_{n-1} + 4b_{n-1} + a_{n-2} + [n = 01]; \\
  b_n &= a_{n-1} + b_{n-1}.
\end{align*}
\]

Hence the generating functions satisfy \( A = 2zA + 4zB + z^2 A + 1, B = zA + zB, \) and we have

\[
A(z) = \frac{1 - z}{(1 + z)(1 - 4z + z^2)}.
\]

This formula relates to the problem of 3 x n domino tilings; we have \( a_n = \frac{1}{3} (U_{2n} + V_{2n+1} + (-1)^n) = \frac{1}{6} (2 + \sqrt{3})^n + \frac{1}{6} (2 - \sqrt{3})^n + \frac{1}{3} (-1)^n \), which is \( (2 + \sqrt{3})^{n+1}/6 \) rounded to the nearest integer.