A ANSWERS TO EXERCISES 545

Subtract $z^{b_m}/(1 - z^{a_m})$ from both sides and set $z = e^{2\pi i/a_m}$. The left side is infinite, and the right side will be finite unless $a_{m-1} = a$.

7.33 \((-1)^{n-m+1}[n > m]/(n - m)\).

7.34 We can also write $G_r(z) = \sum_{k_1+(m+1)k_{m+1} = n} (k_1 + k_{m+1}) (z^m)^{k_{m+1}}$. In general, if

$$G_n = \sum_{k_1 + 2k_2 + \cdots + rk_r = n} \left(\frac{k_1 + k_2 + \cdots + k_r}{k_1, k_2, \ldots, k_r}\right) z_1^{k_1} z_2^{k_2} \cdots z_r^{k_r},$$

we have $G_n = z_1 G_{n-1} + z_2 G_{n-2} + \cdots + z_r G_{n-r} + [n = 0]$, and the generating function is $1/(1 - z_1 w - z_2 w^2 - \cdots - z_r w^r)$. In the stated special case the answer is $1/(1 - w - z^m w^{m+1})$. (See (5.74) for the case $m = 1$.)

7.35 (a) $\frac{1}{n} \sum_{0 < k < n} (1/k + 1/(n-k)) = \frac{2}{n} H_{n-1}$, (b) $[z^n] \left(\ln \frac{1}{1-z}\right)^2 = \frac{2}{n} [z^n]$ = $\frac{2}{n} H_{n-1}$ by (7.50) and (6.58). Another way to do part (b) is to use the rule $[z^n] F(z) = \frac{1}{n} [z^{n-1}] F'(z)$ with $F(z) = (\ln \frac{1}{1-z})^2$.

7.36 $\frac{1}{1-z} A(z^m)$.

7.37 (a) The amazing identity $a_{2n} = a_{2n+1} = b_n$ holds in the table

<table>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
</tr>
<tr>
<td>$b_n$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
</tr>
</tbody>
</table>

(b) $A(z) = 1/(1 - z)(1 - z^2)(1 - z^4)(1 - z^8)\ldots$, (c) $B(z) = A(z)/(1 - z)$, and we want to show that $A(z) = (1 + z)B(z^2)$. This follows from $A(z) = A(z^2)/(1 - z)$.

7.38 \((1 - wz) M(w, z) = \sum_{m,n \geq 1} (\min(m,n) - \min(m-1,n-1)) w^m z^n = \sum_{m,n \geq 1} w^m z^n = wz/(1-w)(1-w)\). In general,

$$M(z_1, \ldots, z_m) = \frac{\prod_{1 \leq i < j \leq m} (1 - z_i) \cdots (1 - z_m)}{(1 - z_1) \cdots (1 - z_m)(1 - z_1 \cdots z_m)}.$$

7.39 The answers to the hint are

$$\sum_{1 \leq i < j < \cdots < k_n \leq n} a_{k_1} a_{k_2} \cdots a_{k_m} \quad \text{and} \quad \sum_{1 \leq i < j < \cdots < k_n \leq n} a_{k_1} a_{k_2} \cdots a_{k_m}.$$

respectively. Therefore: (a) We want the coefficient of $z^m$ in the product $(1 + z)/(1 + 2z)\ldots(1 + nz)$. This is the reflection of $[n+1]^{-1}$, so it is $[n+1]^{-1} + [n+1]^{-1} z + \cdots + [n+1]^{-1} z^n$ and the answer is $[n+1]^{-1}$, (b) The coefficient of $z^m$ in $1/((1 - z)/(1 - 2z)\ldots(1 - nz))$ is $[n+1]^{-1}$ by (7.47).