with the initial conditions
\[ y(t_0) = y_0, \quad y'(t_0) = y'_0. \] (2)

Physical applications often lead to another type of problem, one in which the value of the dependent variable \( y \) or its derivative is specified at two different points. Such conditions are called boundary conditions to distinguish them from initial conditions that specify the value of \( y \) and \( y' \) at the same point. A differential equation together with suitable boundary conditions form a two-point boundary value problem. A typical example is the differential equation
\[ y'' + p(x)y' + q(x)y = g(x) \] (3)
with the boundary conditions
\[ y(\alpha) = y_0, \quad y(\beta) = y_1. \] (4)

The natural occurrence of boundary value problems usually involves a space coordinate as the independent variable so we have used \( x \) rather than \( t \) in Eqs. (3) and (4). To solve the boundary value problem (3), (4) we need to find a function \( y = \phi(x) \) that satisfies the differential equation (3) in the interval \( \alpha < x < \beta \) and that takes on the specified values \( y_0 \) and \( y_1 \) at the endpoints of the interval. Usually, we seek first the general solution of the differential equation and then use the boundary conditions to determine the values of the arbitrary constants.

Boundary value problems can also be posed for nonlinear differential equations but we will restrict ourselves to a consideration of linear equations only. An important classification of linear boundary value problems is whether they are homogeneous or nonhomogeneous. If the function \( g \) has the value zero for each \( x \), and if the boundary values \( y_0 \) and \( y_1 \) are also zero, then the problem (3), (4) is called homogeneous. Otherwise, the problem is nonhomogeneous.

Although the initial value problem (1), (2) and the boundary value problem (3), (4) may superficially appear to be quite similar, their solutions differ in some very important ways. Under mild conditions on the coefficients initial value problems are certain to have a unique solution. On the other hand, boundary value problems under similar conditions may have a unique solution, but may also have no solution or, in some cases, infinitely many solutions. In this respect, linear boundary value problems resemble systems of linear algebraic equations.

Let us recall some facts (see Section 7.3) about the system
\[ Ax = b, \] (5)
where \( A \) is a given \( n \times n \) matrix, \( b \) is a given \( n \times 1 \) vector, and \( x \) is an \( n \times 1 \) vector to be determined. If \( A \) is nonsingular, then the system (5) has a unique solution for any \( b \). However, if \( A \) is singular, then the system (5) has no solution unless \( b \) satisfies a certain additional condition, in which case the system has infinitely many solutions. Now consider the corresponding homogeneous system
\[ Ax = 0, \] (6)
obtained from the system (5) when \( b = 0 \). The homogeneous system (6) always has the solution \( x = 0 \). If \( A \) is nonsingular, then this is the only solution, but if \( A \) is singular, then there are infinitely many (nonzero) solutions. Note that it is impossible for the homogeneous system to have no solution. These results can also be stated in the following way: The nonhomogeneous system (5) has a unique solution if and only