The empty set is pointless.

7.44 Each partition into $k$ nonempty subsets can be ordered in $k!$ ways, so $b_k = k!$. Thus $\mathcal{Q}(z) = \sum_{n,k \geq 0} \binom{n}{k} k! z^n/n! = \sum_{k \geq 0} \binom{n}{k} z^n = 1/(1 - z)$. And this is the geometric series $\sum_{k \geq 0} e^{kz}/2^{k+1}$, hence $a_k = 1/2^{k+1}$. Finally, $c_k = 2^k$; consider all permutations when the $x_i$'s are distinct, change each '<' between subscripts to '>' and allow each '<' between subscripts to become either '<' or '='. (For example, the permutation $x_1 x_2 x_3$ produces $x_1 < x_3 < x_2$ and $x_1 = x_3 < x_2$, because $1 < 3 > 2$.)

7.45 This sum is $\sum_{n \geq 1} r(n)/n^2$, where $r(n)$ is the number of ways to write $n$ as a product of two relatively prime factors. If $n$ is divisible by $t$ distinct primes, $r(n) = 2^t$. Hence $\tau(n)/n^2$ is multiplicative and the sum is

$$\prod_p \left(1 + \frac{2}{p^2} \frac{2}{p^2} \cdots \right) = \prod_p \left(1 + \frac{2}{p^2 - 1} \right) = \prod_p \left(\frac{p^2 + 1}{p^2 - 1} \right) = \zeta(2)/\zeta(4) = \frac{5}{2}.$$

7.46 Let $S_n = \sum_{0 \leq k \leq n/2} \binom{n-2k}{k} \alpha^k$. Then $S_n = S_{n+1} + \alpha S_{n+3} + [n = 0]$, and the generating function is $1/(1 - z - \alpha z^3)$. When $\alpha = -\frac{4}{9}$, the hint tells us that this has a nice factorization $1/(1 + \frac{1}{2} z)(1 - \frac{1}{2} z)^2$. The general expansion theory now yields $S_n = \left(\frac{1}{3} n + c\right)(\frac{1}{3})^n + \frac{1}{3} (-\frac{1}{3})^n$, and the remaining constant $c$ turns out to be $\frac{8}{9}$.

7.47 The Stern-Brocot representation of $\sqrt{3}$ is $\mathbb{R}[(LR^2)^\infty]$, because

$$\sqrt{3} + 1 = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\cdots}}}}.$$

The fractions are $\frac{1}{1}, \frac{1}{1}, \frac{1}{2}, \frac{5}{3}, \frac{1}{4}, \frac{19}{11}, \frac{1}{13}, \ldots$; they eventually have the cyclic pattern

$$\frac{V_{2n-1} + V_{2n+1}}{U_{2n}}, \frac{U_{2n} + V_{2n+1}}{V_{2n+1}}, \frac{U_{2n+2} + V_{2n+1}}{U_{2n+1}}, \frac{V_{2n+1} + V_{2n+3}}{U_{2n+2}}, \ldots.$$

7.48 We have $g_0 = 0$, and if $g_1 = m$ the generating function satisfies

$$\alpha G(z) + bz^{-1} G(z) + cz^{-2}(G(z) - mz) + \frac{d}{1-z} = 0.$$

Hence $G(z) = P(z)/(az^2 + bz + c)(1 - z)$ for some polynomial $P(z)$. Let $\rho_1$ and $\rho_2$ be the roots of $cz^2 + bz + a$, with $|\rho_1| > |\rho_2|$. If $b^2 - 4ac < 0$ then $|\rho_1|^2 = \rho_1 \rho_2 = a/c$ is rational, contradicting the fact that $\sqrt{g_n}$ approaches