548 ANSWERS TO EXERCISES

1 + \sqrt{2}. Hence \( p_1 = (\sqrt{b^2 - 4ac} - b) / 2c = 1 + \sqrt{2} \), and this implies that \( a = -c, b = -2c, p_2 = 1 - \sqrt{2} \). The generating function now takes the form

\[
G(z) = \frac{z(m - (r + m)z)}{(1 - 2z - z^2)(1 - z)}
\]

where \( r = d/c \). Since \( g_2 \) is an integer, \( r \) is an integer. We also have

\[
g_n = a(1 + \sqrt{2})^n + \alpha(1 - \sqrt{2})^n + \frac{1}{2}r = \frac{\alpha(1 + \sqrt{2})^n}{2},
\]

and this can hold only if \( r = -1 \), because \((1 - \sqrt{2})^n\) alternates in sign as it approaches zero. Hence (a, b, c, d) = \pm (1, 2, -1, 1). Now we find \( \alpha = \frac{1}{4} (1 + \sqrt{2} m) \), which is between 0 and 1 only if \( 0 \leq m \leq 2 \). Each of these values actually gives a solution; the sequences \( \langle g_n \rangle \) are \( \langle 0, 0, 1, 3, 8, \ldots \rangle \), \( \langle 0, 1, 3, 8, 20, \ldots \rangle \), and \( \langle 0, 2, 5, 13, 32, \ldots \rangle \).

7.49 (a) The denominator of \((1/(1 - \sqrt{2}z)) + 1/(1 - (1 + \sqrt{2})z)\) is \( 1 - 2z - z^2 \); hence \( a_n = 2a_{n-1} + a_{n-2} \) for \( n \geq 2 \). (b) True because \( a_n \) is even and \(-1 < 1 - \sqrt{2} < 0\). (c) Let

\[
b_n = \left( \frac{p + \sqrt{q}}{2} \right)^n + \left( \frac{p - \sqrt{q}}{2} \right)^n
\]

We would like \( b_n \) to be odd for all \( n > 0 \), and \(-1 < (p + \sqrt{q})/2 < 0\). Working as in part (a), we find \( b_0 = 2, b_1 = p, and b_n = pb_{n-1} + \frac{1}{4}(p^2 - q^2)b_{n-2} \) for \( n \geq 2 \). One satisfactory solution has \( p = 3 \) and \( q = 17 \).

7.50 Extending the multiplication idea of exercise 22, we have

\[
Q = Q + Q + Q + Q + Q + Q + \cdots
\]

Replace each n-gon by \( z^{n-2} \). This substitution behaves properly under multiplication, because the pasting operation takes an m-gon and an n-gon into an \( (m + n - 2) \)-gon. Thus the generating function is

\[
Q = 1 + zQ^2 + z^2Q^3 + z^4Q^4 + \cdots = 1 + \frac{zQ^2}{1 - zQ}
\]

and the quadratic formula gives \( Q = (1 + z - \sqrt{1 - 6z + z^2}) / 2z \). The coefficient of \( z^n \) in this power series is the number of ways to put nonoverlapping diagonals into a convex n-gon. These coefficients apparently have no closed form in terms of other quantities that we have discussed in this book, but their asymptotic behavior is known [173, exercise 2.2.1-12].

Give me Legendre polynomials and I’ll give you a closed form.