Incidentally, if each n-gon in Q is replaced by $wz^{n-2}$ we get
\[ Q = \frac{1 + z - \sqrt{1 - (4w + 2)z + z^2}}{2(1 + w)z}, \]
a formula in which the coefficient of $w^m z^{n-2}$ is the number of ways to divide an n-gon into m polygons by nonintersecting diagonals.

7.51 The key first step is to observe that the square of the number of ways is the number of cycle patterns of a certain kind, generalizing exercise 27. These can be enumerated by evaluating the determinant of a matrix whose eigenvalues are not difficult to determine. When $m = 3$ and $n = 4$, the fact that $\cos 36^\circ = \phi/2$ is helpful (exercise 6.46).

7.52 The first few cases are $p_0(y) = 1$, $p_1(y) = y$, $p_2(y) = y^2 + y$, $p_3(y) = y^3 + 3y^2 + 3y$. Let $p_i(y) = q_{2n}(x)$ where $y = x(1 \ x)$; we seek a generating function that defines $q_{2n+1}(x)$ in a convenient way. One such function is $\sum q_n(x) z^n/n! = 2e^{ix}/(e^{iz} + 1)$, from which it follows that $q_n(x) = i^n E_n(x)$, where $E_n(x)$ is called an Euler polynomial. We have $\sum (-1)^k x^n \delta x = \frac{i}{2} (-1)^{n+1} E_n(x)$, so Euler polynomials are analogous to Bernoulli polynomials, and they have factors analogous to those in (6.98). By exercise 6.23 we have $n E_{n-1}(x) = \sum_{k=0}^n \binom{n}{k} B_k x^k(2 - 2^{k+1})$; this polynomial has integer coefficients by exercise 6.54. Hence $q_{2n}(x)$, whose coefficients have denominators that are powers of 2, must have integer coefficients. Hence $p_i(y)$ has integer coefficients. Finally, the relation $(4y - 1)p_n''(y) + 2p_n'(y) = 2n(2n - 1)p_{n-1}(y)$ shows that
\[ 2m(2m - 1) \frac{n}{m} = m(m + 1) \frac{n}{m} + 2n(2n - 1) \frac{n - 1}{m - 1}, \]
and it follows that the $\frac{n}{m}$'s are positive. (A similar proof shows that the related quantity $(-1)^n(2n + 2)E_{2n-1}(x)/(2x - 1)$ has positive integer coefficients, when expressed as an nth degree polynomial in y.) It can be shown that $\frac{n}{m} = \binom{n}{m}$ is the Genocchi number $(-1)^{n-1}(2^{n+1} - 2)|B_{2n}|$ (see exercise 6.24), and that $\frac{n}{m} = \binom{n}{m}$ is $2^{(n+1)/2} + 3\binom{n}{3}$, etc.

7.53 It is $P_{1+V_{k-1}+V_{k-2}/6}$. Thus, for example, $T_{20} = P_{12} = 210; T_{285} = P_{165} = 40755$.

7.54 Let $E_k$ be the operation on power series that sets all coefficients to zero except those of $z^n$ where $n \mod m = k$. The stated construction is equivalent to the operation
\[ E_0 S E_0 S (E_0 + E_1) S \ldots S (E_0 + E_1 + \cdots + E_{m-1}) \]