This distribution has $\kappa_4 = 2974$, which is rather large. Hence the standard deviation of this variance estimate when $n = 10$ is also rather large, $\sqrt{2974/10 + 2(22)^2/9} \approx 20.1$ according to exercise 54. One cannot complain that the students cheated.

8.4 This follows from (8.38) and (8.39), because $F(z) = G(z)H(z)$. (A similar formula holds for all the cumulants, even though $F(z)$ and $G(z)$ may have negative coefficients.)

8.5 Replace $H$ by $p$ and $T$ by $q = \frac{1}{2}$. If $S_A = S_B = \frac{1}{2}$ we have $p^2 q N = \frac{1}{2}$ and $pq^2 N = \frac{1}{4} q + \frac{1}{2} j$; the solution is $p = 1/\sqrt{2}$, $q = 1/\sqrt{2}$.

8.6 In this case $X/Y$ has the same distribution as $X$, for all $y$, hence $E(X/Y) = EX$ is constant and $V(E(X/Y)) = 0$. Also $V(X|Y)$ is constant and equal to its expected value.

8.7 We have $\mathbf{1} = (p_1 + p_2 + \ldots + p_6)^2 \leq \sum (p_1^2 + p_2^2 + \ldots + p_6^2)$ by Chebyshev's summation inequality of Chapter 2.

8.8 Let $p = \Pr(\omega \in A \cap B)$, $q = \Pr(\omega \notin A)$, and $r = \Pr(\omega \notin B)$. Then $p + q + r = 1$, and the identity to be proved is $\rho_n = (p + r)(p + q) - qr$.

8.9 This is true (subject to the obvious proviso that $F$ and $G$ are defined on the respective ranges of $X$ and $Y$), because

$$\Pr(F(X) = f \text{ and } G(Y) = g) = \sum_{x \in F^{-1}(f) \cap G^{-1}(g)} \Pr(X = x \text{ and } Y = y)$$

$$= \sum_{x \in F^{-1}(f) \cap G^{-1}(g)} \Pr(X = x) \cdot \Pr(Y = y)$$

$$= \Pr(F(X) = f) \cdot \Pr(G(Y) = g).$$

8.10 Two. Let $x_1 < x_2$ be medians; then $\mathbf{1} \leq \Pr(X \leq x_1) + \Pr(X \geq x_2) \leq 1$, hence equality holds. (Some discrete distributions have no median elements. For example, let $\Omega$ be the set of all fractions of the form $\pm 1/n$, with $\Pr(\pm 1/n) = \frac{1}{12} n^{-2}$.)

8.11 For example, let $K = k$ with probability $4/(k + 1) (k + 2) (k + 3)$, for all integers $k \geq 0$. Then $\mathbb{E}K = 1$, but $\mathbb{E}[K^2] = \infty$. (Similarly we can construct random variables with finite cumulants through $\kappa_m$ but with $\kappa_{m+1} = \infty$.)

8.12 (a) Let $p_k = \Pr(X = k)$. If $0 < x \leq 1$, we have $\Pr(X \leq x) = \sum_{k \leq r} p_k \leq \sum_{k \leq r} x^{k-r} p_k \leq \sum_{k \leq r} x^{k-r} p(x) = x^{-r} p(x)$. The other inequality has a similar proof. (b) Let $x = \alpha/(1 - \alpha)$ to minimize the right-hand side. (A more precise estimate for the given sum is obtained in exercise 9.42.)