8.19 (a) $G_{X_1+X_2}(z) = G_{X_1}(z) G_{X_2}(z) = e^{(\mu_1 + \mu_2)z}$. Hence the probability is $e^{(\mu_1 + \mu_2)z}/n!$; the sum of independent Poisson variables is Poisson.

(b) In general, if $K_nX$ denotes the $n$th cumulant of a random variable $X$, we have $K_n(aX_1 + bX_2) = a^nK_nX_1 + b^mK_nX_2$, when $a, b \geq 0$. Hence the answer is $2^m\mu_1 + 3^m\mu_2$.

8.20 The general pgf will be $G(z) = z^m/F(z)$, where

$$F(z) = z^m - (1 - z) \sum_{k=1}^{m} \tilde{A}_{[k]} [A^{(k)} = A_{[k]}] z^{m-k},$$

$$F'(1) = m - \sum_{k=1}^{m} \tilde{A}_{[k]} [A^{(k)} = A_{[k]}],$$

$$F''(1) = m(m-1) - 2 \sum_{k=1}^{m} (m-k) \tilde{A}_{[k]} [A^{(k)} = A_{[k]}].$$

8.21 This is $\sum_{n=0}^{\infty} q_n$, where $q_n$ is the probability that the game between Alice and Bill is still incomplete after $n$ flips. Let $p_n$ be the probability that the game ends at the $n$th flip; then $p_n + q_n = q_{n-1}$. Hence the average time to play the game is $\sum_{n=1}^{\infty} np_n = (q_0 - q_1) + 2(q_1 - q_2) + 3(q_2 - q_3) + \cdots = q_0 + q_1 + q_2 + \cdots = N$, since $\lim_{n \to \infty} np_n = 0$.

Another way to establish this answer is to replace $H$ and $T$ by $\frac{1}{2}z$. Then the derivative of the first equation in (8.78) tells us that $N(1) + N'(1) = N'(1) + S^1(1) + S^1_0(1)$. By the way, $N = \frac{16}{3}$.

8.22 By definition we have $V(X|Y) = E[X^2|Y] - (E[X|Y])^2$ and $V(E(X|Y)) = E[(E(X|Y))^2] - (E(E(X|Y)))^2$; hence $E(V(X|Y)) + V(E(X|Y)) = E(E(X^2|Y)) - (E(E(X|Y)))^2$. But $E(E(X|Y)) = EX$ and $E(E(X^2|Y)) = E(X^2)$, so the result is just $EX$.

8.23 Let $\Omega_0 = \{\emptyset, [.]\}$ and $\Omega_1 = \{[.]', [.]''\}$; and let $\Omega_2$ be the other 16 elements of $\Omega$. Then $P_{[1]}(w) = P_{[00]}(w) = \frac{2+6}{576}, \frac{2}{576}, \frac{2}{576}$ according as $w \in \Omega_0, \Omega_1, \Omega_2$. The events $A$ must therefore be chosen with $k_i$ elements from $\Omega_i$, where $(k_0, k_1, k_2)$ is one of the following: $(0, 0, 0), (0, 2, 7), (0, 4, 14), (1, 4, 4), (1, 6, 1), (2, 6, 1), (2, 8, 8), (3, 8, 15), (3, 10, 5), (3, 12, 12), (4, 12, 2), (4, 14, 9), (4, 16, 16)$. Foreexample, there are $\binom{1}{3} \binom{16}{16}$ events of type $(2, 6, 1)$. The total number of such events is $[z^2] (1 + z^2)^4 (1 + z^3)^2 (1 + z^2)^{16}$, which turns out to be $1304927002$. If we restrict ourselves to events that depend on $S$ only, we get 40 solutions $S \in A$, where $A = \emptyset, \{2, 4, 6, 8, 10, 12\}, \{2, 6, 8, 10, 12\}, \{3, 7, 9, 11, 12\}$, and the complements of these sets. (Here the notation $\{2, 4, 6, 8, 10, 12\}$ means either 2 or 12 but not both.)