A ANSWERS TO EXERCISES 555

\[ \frac{17}{3} \]; in case (b) it is \( \frac{14}{3} \). Bill wins only half as often, but when he does win he tends to win sooner. The overall average number of flips is \( \frac{17}{3} + \frac{14}{3} = \frac{31}{3} \), agreeing with exercise 21. The solitaire game for each pattern has a waiting time of 8.

8.29 Set \( H = T = \frac{1}{2} \) in

\[
1 + N[H + T] = N + S_A + S_B + S_C
\]

\[
N H H T H = S_A (1 + HTH) + S_B (HTH + TH + 1) + S_C (HTH + TH)
\]

\[
N H T H H = S_A (THH + H) + S_B (THH + 1) + S_C (THH)
\]

\[
N T H H H = S_A (HH) + S_B (HH) + S_C
\]

to get the winning probabilities. In general we will have \( S_A + S_B + S_C = 1 \) and

\[
S_A (A:A) + S_B (B:B) + S_C (C:C) = S_A (A:B) + S_B (B:C) + S_C (C:B)
\]

In particular, the equations \( 9S_A + 3S_B + 3S_C = 5S_A + 9S_B + S_C = 2S_A + 4S_B + 9S_C \) imply that \( S_A = \frac{16}{52}, S_B = \frac{17}{52}, S_C = \frac{19}{52} \).

8.30 The variance of \( P[h_1, \ldots, h_n; k] \) is the variance of the shifted binomial distribution \( ((m - 1 + z)/m)^k \) which is \( (k-1)(\frac{1}{m})[1 - \frac{1}{m}] \) by (8.61).

Hence the average of the variance is \( \text{Mean}(S)(m - 1)/m^2 \). The variance of the average is the variance of \( (k - 1)/m \), namely \( \text{Var}(S)/m^2 \). According to (8.105), the sum of these two quantities should be \( VP \), and it is. Indeed, we have just replayed the derivation of (8.95) in slight disguise. (See exercise 15.)

8.31 (a) A brute force solution would set up five equations in five unknowns:

\[
A = \frac{1}{2} zB + \frac{1}{2} zE; \quad B = \frac{1}{2} zC; \quad C = 1 + \frac{1}{2} zB + \frac{1}{2} zD; \quad D = \frac{1}{2} zC + \frac{1}{2} zE; \quad E = \frac{1}{2} zD.
\]

But positions \( C \) and \( D \) are equidistant from the goal, as are \( B \) and \( E \), so we can lump them together. If \( X = B + E \) and \( Y = C + D \), there are now three equations:

\[
A = \frac{1}{2} zX; \quad X = \frac{1}{2} zY; \quad Y = 1 + \frac{1}{2} zX + \frac{1}{2} zY.
\]

Hence \( A = z^2/(4 - 2z^2) \); we have \( \text{Mean}(A) = 6 \) and \( \text{Var}(A) = 22 \). (Rings a bell? In fact, this problem is equivalent to flipping a fair coin until getting heads twice in a row: Heads means “advance toward the apple” and tails means “go back.”) (b) Chebyshev’s inequality says that \( \text{Pr}(S \geq 100) \leq \text{Pr}(S \geq 94^2) \leq 94^2/22 \approx .0025 \). (c) The second tail equality says that \( \text{Pr}(S \geq 100) \leq 1/\chi^2(4 - 2x - x^2) \) for all \( x \geq 1 \), and we get the upper bound approximately \( 0.0000000009 \) when \( x = (\sqrt{99} - 99)/100 \). (The actual probability is approximately \( 0.0000000009 \), according to exercise 37.)