By writing the constant term in Eq. (9) as \( a_0/2 \), it is possible to compute all the \( a_n \) from Eq. (13). Otherwise, a separate formula would have to be used for \( a_0 \).

A similar expression for \( b_n \) may be obtained by multiplying Eq. (9) by \( \sin(n\pi x/L) \), integrating termwise from \(-L\) to \(L\), and using the orthogonality relations (7) and (8); thus

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \ldots
\]  

Equations (13) and (14) are known as the Euler–Fourier formulas for the coefficients in a Fourier series. Hence, if the series (9) converges to \( f(x) \), and if the series can be integrated term by term, then the coefficients must be given by Eqs. (13) and (14).

Note that Eqs. (13) and (14) are explicit formulas for \( a_n \) and \( b_n \) in terms of \( f \), and that the determination of any particular coefficient is independent of all the other coefficients. Of course, the difficulty in evaluating the integrals in Eqs. (13) and (14) depends very much on the particular function \( f \) involved.

Note also that the formulas (13) and (14) depend only on the values of \( f(x) \) in the interval \(-L \leq x \leq L\). Since each of the terms in the Fourier series (9) is periodic with period 2\(L\), the series converges for all \(x\) whenever it converges in \(-L \leq x \leq L\), and its sum is also a periodic function with period 2\(L\). Hence \( f(x) \) is determined for all \(x\) by its values in the interval \(-L \leq x \leq L\).

It is possible to show (see Problem 27) that if \( g \) is periodic with period \( T \), then every integral of \( g \) over an interval of length \( T \) has the same value. If we apply this result to the Euler–Fourier formulas (13) and (14), it follows that the interval of integration, \(-L \leq x \leq L\), can be replaced, if it is more convenient to do so, by any other interval of length 2\(L\).

**Example 1**

Assume that there is a Fourier series converging to the function \( f \) defined by

\[
f(x) = \begin{cases} 
-x, & -2 \leq x < 0, \\
x, & 0 \leq x < 2;
\end{cases}
\]

\( f(x + 4) = f(x) \).

Determine the coefficients in this Fourier series.

This function represents a triangular wave (see Figure 10.2.2) and is periodic with period 4. Thus in this case \( L = 2 \) and the Fourier series has the form

\[
f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos\left(\frac{m\pi x}{2}\right) + b_m \sin\left(\frac{m\pi x}{2}\right) \right),
\]

where the coefficients are computed from Eqs. (13) and (14) with \( L = 2 \). Substituting for \( f(x) \) in Eq. (13) with \( m = 0 \), we have

\[
a_0 = \frac{1}{2} \int_{-2}^{0} (-x) \, dx + \frac{1}{2} \int_{0}^{2} x \, dx = 1 + 1 = 2.
\]

For \( m > 0 \), Eq. (13) yields

\[
a_m = \frac{1}{2} \int_{-2}^{0} (-x) \cos\left(\frac{m\pi x}{2}\right) \, dx + \frac{1}{2} \int_{0}^{2} x \cos\left(\frac{m\pi x}{2}\right) \, dx.
\]