The solution is given by the super generating function

\[ G(w, z) = \sum_{n \geq 0} G_n(z)w^n = A(w)/(1 - zB(w)) , \]

where \( B(w) = w(qf - (qf - ph)w)/(1 - qhw) \) and \( A(w) = (1 - B(w))/(1 - w) \).

Now \( \sum_{n \geq 0} G_n'(1)w^n = \alpha w/(1 - w)^2 + \beta/(1 - w) = \beta/(1 - (qf - ph)w) \) where

\[ \alpha = \frac{ph}{ph + pf} , \quad \beta = \frac{pf(qf - ph)}{(ph + pf)^2} , \]

hence \( G_n'(1) = \alpha n + \beta (1 - (qf - ph)n) \). (Similarly \( G''_n(1) = \alpha^2 n^2 + O(n) \), so the variance is \( O(n) \).

8.43 \( G_n(z) = \sum_{k \geq 0} \binom{n}{k} z^k/n! = z^n/n!, \) by (6.11). This is a product of binomial pgf’s, \( \prod_{k=1}^{n} ((k-1 + z)/k) \), where the kth has mean \( 1/k \) and variance \( (k-1)/k^2 \); hence \( \text{Mean}(G_n) = H_n \) and \( \text{Var}(G_n) = H_n - H_n^{(2)} \).

8.44 (a) The champion must be undefeated in \( n \) rounds, so the answer is \( p^n \).

(b) Players \( x_1, \ldots, x_2^n \) must be “seeded” (by chance) in distinct subtournaments and they must win all \( 2^n(n - k) \) of their matches. The \( 2^n \) leaves of the tournament tree can be filled in \( 2^n! \) ways; to seed it we have \( 2^k!(2^n - k)^k \) ways to place the top \( 2^k \) players, and \( (2^n - 2^k)! \) ways to place the others. Hence the probability is \( (2p)^k n^{n-k}/(2^n - 1) \). (d) Each tournament outcome corresponds to a permutation of the players: Let \( y_1 \) be the champ; let \( y_2 \) be the other finalist; let \( y_3 \) and \( y_4 \) be the players who lost to \( y_1 \) and \( y_2 \) in the semifinals; let \( \{y_5, \ldots, y_8\} \) be those who lost respectively to \( \{y_1, \ldots, y_4\} \) in the quarterfinals; etc. (Another proof shows that the first round has \( 2^n!/2^{n-1}! \) essentially different outcomes; the second round has \( 2^{n-1}!/2^{n-2}! \); and so on.) (e) Let \( S_k \) be the set of \( 2^{k-1} \) potential opponents of \( x_k \) in the \( k \)th round. The conditional probability that \( x_2 \) wins, given that \( x_1 \) belongs to \( S_k \), is

\[ \Pr(x_1 \text{ plays } x_2) \cdot p^{n-1}(1 - p) + \Pr(x_1 \text{ doesn’t play } x_2) \cdot p^n \]

\[ = p^{k-1}p^{n-1}(1 - p) + (1 - p^{k-1})p^n . \]

The chance that \( x_1 \in S_k \) is \( 2^{k-1}/(2^n - 1) \); summing on \( k \) gives the answer:

\[ \sum_{k=1}^{n} \frac{2^{k-1}}{2^n - 1} \left( p^{k-1}p^{n-1}(1 - p) + (1 - p^{k-1})p^n \right) = p^n - \frac{(2p)^n - 1}{2^n - 1}p^{n-1} . \]

(f) Each of the \( 2^n! \) tournament outcomes has a certain probability of occurring, and the probability that \( x_i \) wins is the sum of these probabilities over all \( (2^n - 1)! \) tournament outcomes in which \( x_i \) is champion. Consider interchanging \( x_i \) with \( x_{i+1} \) in all those outcomes; this change doesn’t affect the