\(X_n + Y_n\), where \(Y_n = -1\) if the \((n + 1)\text{st}\) particle hits a diphage receptor (conditional probability \(2X_n/(n + 2)\)) and \(Y_n = +2\) otherwise. Hence

\[
E^{\text{X}_{n+1}} = EX_n + EY_n = EX_n + 2EX_n/(n+2) + 2(1 - 2EX_n/(n+2)).
\]

The recurrence \((n+2)E^{\text{X}_{n+1}} = (n-4)E^{\text{X}_{n+2}} + 2n+4\) can be solved if we multiply both sides by the summation factor \((n + 1)\); or we can guess the answer and prove it by induction: \(EX_n = (2n + 4)/7\) for all \(n > 4\). (Incidentally, there are always two diphages and one triphage after five steps, regardless of the configuration after four.)

8.48 (a) The distance between frisbees (measured so as to make it an even number) is either 0, 2, or 4 units, initially 4. The corresponding generating functions \(A, B, C\) (where, say, \([z^n]\) \(C\) is the probability of distance 4 after \(n\) throws) satisfy

\[
A = \frac{1}{4}zB, \quad B = \frac{1}{2}zB + \frac{1}{4}zC, \quad C = 1 + \frac{1}{4}zB + \frac{3}{4}zC
\]

It follows that \(A = z^{2/16} - 202 + 5z^2 = z^2/F(z)\), and we have \(\text{Mean}(A) = 2\text{ Mean}(F) = 12, \text{ Var}(A) = \text{ Var}(F) = 100.\) (A more difficult but more amusing solution factors \(A\) as follows:

\[
A = \frac{p_1 z}{1 - q_1 z} + \frac{p_2 z}{1 - q_2 z} = \frac{p_1 z}{p_2 - p_1 1 - q_1 z} + \frac{p_1 p_2 z}{p_1 - p_2 1 - q_2 z},
\]

where \(p_1 = \phi^2/4 = (3 + \sqrt{5})/8, p_2 = \phi^2/4 = (3 - \sqrt{5})/8, p_1 + q_1 = p_2 + q_2 = 1.\) Thus, the game is equivalent to having two biased coins whose heads probabilities are \(p_1\) and \(p_2\); flip the coins one at a time until they have both come up heads, and the total number of flips will have the same distribution as the number of frisbee throws. The mean and variance of the waiting times for these two coins are respectively \(6 \div 2\sqrt{5}\) and \(50 \div 22\sqrt{5}\), hence the total mean and variance are 12 and 100 as before.)

(b) Expanding the generating function in partial fractions makes it possible to sum the probabilities. (Note that \(\sqrt{5}/(4\phi) + \phi^2/4 = 1\), so the answer can be stated in terms of powers of \(\phi\).) The game will last more than \(n\) steps with probability \(5^{n+1}/2^{n+2} - n F_{n+2}\); when \(n\) is even this is \(5^{n/2} F_{n+2}\). So the answer is \(5^{50.4} 10^{8} F_{102} \approx .00006.\)

8.49 (a) If \(n > 0\), \(P_N(0, n) = \frac{1}{2}[N = 0] + \frac{1}{4}P_{N-1}(0, n) + \frac{1}{4}P_{N-1}(1, n-1); P_N(0, 0) = [N = 0].\) Hence

\[
g_{m, n} = \frac{1}{2}zg_{m, n+1} + \frac{1}{2}zg_{m, n} + \frac{1}{4}zg_{m+1, n}; \quad g_{0, n} = \frac{1}{2} + \frac{1}{4}zg_{0, n} + \frac{1}{4}g_{1, n-1}; \quad \text{etc.}
\]

(b) \(g'_{m, n} = 1 + \frac{1}{2}g'_{m, n+1} + \frac{1}{2}g'_{m, n} + \frac{1}{4}g'_{m+1, n-1}; g'_{0, n} = \frac{1}{2} + \frac{1}{4}g'_{0, n} + \frac{1}{4}g'_{1, n-1}; \text{etc.}\) By induction on \(m\), we have \(g'_{m, n} = (2m + 1)g'_{0, m+n}\) for all \(m, n \geq 0.\)