And since $g'_{m,0} = g_0'$, we must have $g'_{m,n} = m + n + 2mn$. (c) The recurrence is satisfied when $mn > 0$, because

$$\sin(2m + 1)\theta = \frac{1}{\cos^2\theta} \left( \frac{\sin(2m - 1)\theta}{4} + \frac{\sin(2m + 1)\theta}{2} + \frac{\sin(2m + 3)\theta}{4} \right),$$

this is a consequence of the identity $\sin(x - y) + \sin(x + y) = 2\sin x \cos y$. So all that remains is to check the boundary conditions.

8.50 (a) Using the hint, we get

$$3(1 - z)^2 \sum_k \left(\frac{1/2}{k}\right) \left(\frac{8}{9} z\right)^k (1 - z)^2 k$$

$$= 3(1 - z)^2 \sum_k \left(\frac{1/2}{k}\right) \left(\frac{8}{9}\right)^k \sum_{i=1}^k \binom{k + j - 3}{j} z^{j+k};$$

now look at the coefficient of $z^{j+1}$.

(b) $H(z) = \frac{2}{3} + \frac{2}{9} z + \frac{1}{3} \sum_{i=0}^1 c_{3+i} z^{2+i}$.

(c) Let $r = \sqrt{(1-2)(9-z)}$. We can show that $(z-3+r)(z-3-r) = 4z$, and hence that $\left(\frac{r}{1-z}\right)^2 = \frac{(13 - 5z + 4r)/(1-z) = (9 - H(z))/(1-H(z))}.$

(d) Evaluating the first derivative at $z = 1$ shows that $\text{Mean}(H) = 1$. The second derivative diverges at $z = 1$, so the variance is infinite.

8.51 (a) Let $H_n(z)$ be the pgf for your holdings after $n$ rounds of play, with $H_0(z) = z$. The distribution for $n$ rounds is

$$H_{n+1}(z) = H_n\left(H(z)\right),$$

so the result is true by induction (using the amazing identity of the preceding problem). (b) $g_n = H_n(0) - H_{n-1}(0) = 4/n(n+1)(n+2) = 4(n-1)/3$. The mean is 2, and the variance is infinite. (c) The expected number of tickets you buy on the $n$th round is $\text{Mean}(H_n) = 1$, by exercise 15. So the total expected number of tickets is infinite. (Thus, you almost surely lose eventually, and you expect to lose after the second game, yet you also expect to buy an infinite number of tickets.) (d) Now the pgf after $n$ games is $H_n(z)^2$, and the method of part (b) yields a mean of $16 - \frac{4}{3} \pi^2 \approx 2.8$. (The sum $\sum_{k=1}^\infty 1/k^2 = \pi^2/6$ shows up here.)

8.52 If $\omega$ and $\omega'$ are events with $\Pr(\omega) > \Pr(\omega')$, then a sequence of $n$ independent experiments will encounter $\omega$ more often than $\omega'$, with high probability, because $\omega$ will occur very nearly $n\Pr(\omega')$ times. Consequently, as $n \to \infty$, the probability approaches 1 that the median or mode of the