values of \( X \) in a sequence of independent trials will be a median or mode of the random variable \( X \).

8.53 We can disprove the statement, even in the special case that each variable is 0 or 1. Let \( p_0 = \Pr(X = Y = Z = 0) \), \( p_1 = \Pr(X = Y = Z = 1) \), \( p_7 = \Pr(X = Y = Z = 0) \), where \( X = 1 - X \). Then \( p_0 + p_1 + \cdots + p_7 = 1 \), and the variables are independent in pairs if and only if we have

\[
(p_0 + p_5 + p_6 + p_7)(p_2 + p_3 + p_6 + p_7) = p_6 + p_7,
\]
\[
(p_4 + p_5 + p_6 + p_7)(p_1 + p_3 + p_5 + p_7) = p_5 + p_7,
\]
\[
(p_2 + p_3 + p_6 + p_7)(p_1 + p_3 + p_5 + p_7) = p_3 + p_7.
\]
But \( \Pr(X + Y = Z = 0) \neq \Pr(X + Y = 0) \Pr(Z = 0) \iff p_0 \neq (p_0 + p_1)(p_0 + p_2 + p_4 + p_6) \). One solution is

\[ p_0 = p_5 = p_6 = 1/4; \quad p_1 = p_2 = p_4 = p_7 = 0. \]

This is equivalent to flipping two fair coins and letting \( X \) = (the first coin is heads), \( Y \) = (the second coin is heads), \( Z \) = (the coins differ). Another example, with all probabilities nonzero, is

\[ p_0 = 4/64, \quad p_1 = p_2 = p_4 = 5/64, \]
\[ p_3 = p_5 = p_6 = 10/64, \quad p_7 = 15/64. \]

For this reason we say that \( n \) variables \( X_1, \ldots, X_n \) are independent if

\[ \Pr(X_1 = x_1 \text{ and } \cdots \text{ and } X_n = x_n) = \Pr(X_1 = x_1) \cdots \Pr(X_n = x_n); \]

pairwise independence isn’t enough to guarantee this.

8.54 (See exercise 27 for notation.) We have

\[
E(\Sigma_1^2) = n\mu_4 + n(n-1)\mu_2^2;
\]
\[
E(\Sigma_2^2) = n\mu_4 + 2n(n-1)\mu_3\mu_1 + n(n-1)\mu_2^2 + n(n-1)(n-2)\mu_2\mu_1^2;
\]
\[
E(\Sigma_4^4) = n\mu_4 + 4n(n-1)\mu_3\mu_1 + 3n(n-1)\mu_2^2
\]
\[
+ 6n(n-1)(n-2)\mu_2\mu_1^2 + n(n-1)(n-2)(n-3)\mu_1^4;
\]

it follows that \( V(\hat{X}) = k_4/n + 2k_2^2/(n - 1) \).

8.55 There are \( A = \frac{1}{17} \cdot 52! \) permutations with \( X = Y \), and \( B = \frac{16}{17} \cdot 52! \) permutations with \( X \neq Y \). After the stated procedure, each permutation with \( X = Y \) occurs with probability \( \frac{1}{17}/((1 - \frac{16}{17})p)A \), because we return to step \( S_1 \) with probability \( \frac{1}{17}p \). Similarly, each permutation with \( X \neq Y \) occurs with probability \( \frac{16}{17}((1 - p)/((1 - \frac{16}{17})p)B \). Choosing \( p = \frac{1}{4} \) makes \( \Pr(X = x \text{ and } Y = y) = \frac{1}{159} \) for all \( x \) and \( y \) (We could therefore make two flips of a fair coin and go back to \( S_1 \) if both come up heads.)