8.56 If \( m \) is even, the frisbees always stay an odd distance apart and the game lasts forever. If \( m = 2l + 1 \), the relevant generating functions are

\[
G_m = \frac{1}{4} z A_1, \\
A_1 = \frac{1}{4} z A_1 + \frac{1}{4} z A_2, \\
A_k = \frac{1}{4} z A_{k-1} + \frac{1}{4} z A_k + \frac{1}{4} z A_{k+1}, \quad \text{for} \ 1 < k < l, \\
A_l = \frac{1}{4} z A_{l-1} + \frac{1}{4} z A_l + 1.
\]

(The coefficient \( [z^n] A_k \) is the probability that the distance between frisbees is \( 2k \) after \( n \) throws.) Taking a clue from the similar equations in exercise 49, we set \( z = 1 / \cos^2 \theta \) and \( A_1 = X \sin 2\theta \), where \( X \) is to be determined. It follows by induction (not using the equation for \( A_1 \)) that \( A_k = X \sin 2k\theta \). Therefore we want to choose \( X \) such that

\[
\left( 1 - \frac{3}{4 \cos^2 \theta} \right) X \sin 2l\theta = 1 + \frac{1}{4 \cos^2 \theta} X \sin (2l - 2)\theta.
\]

It turns out that \( X = 2 \cos^2 \theta / \sin \theta \cos (2l + 1)\theta \), hence

\[
G_m = \frac{\cos e}{\cos m \theta}.
\]

The denominator vanishes when \( \theta \) is an odd multiple of \( \pi/(2m) \); thus \( 1 - q_k z \) is a root of the denominator for \( 1 \leq k \leq \frac{1}{2} \) and the stated product representation must hold. To find the mean and variance we can write

\[
G_m = \frac{[1 - \frac{1}{2} \theta^2 + \frac{1}{24} \theta^4 - \cdots]}{[1 - \frac{1}{2} m^2 \theta^2 + \frac{1}{24} m^4 \theta^4 - \cdots]} = 1 + \frac{1}{2} (m^2 - 1) \theta^2 + \frac{1}{24} (5m^4 - 6m^2 + 1) \theta^4 + \cdots
\]

because \( \tan^2 \theta = z - 1 \) and \( \tan \theta = \theta + \frac{1}{3} \theta^3 + \cdots \). So we have Mean\( (G_m) = \frac{1}{2} (m^2 - 1) \) and Var\( (G_m) = \frac{1}{6} m^2 (m^2 - 1) \). (Note that this implies the identities

\[
\frac{m^2 - 1}{2} = \sum_{k=1}^{\frac{m-1}{2}} \frac{1}{p_k} = \sum_{k=1}^{\frac{m-1}{2}} \left( \frac{1}{\sin \left( \frac{(2k-1)\pi}{2m} \right)} \right)^2;
\]

\[
\frac{m^2(m^2 - 1)}{6} = \sum_{k=1}^{\frac{m-1}{2}} \left( \cot \frac{(2k-1)\pi}{2m} / \sin \left( \frac{(2k-1)\pi}{2m} \right) \right)^2.
\]

The third cumulant of this distribution is \( \frac{1}{30} m^2 (m^2 - 1)(4m^2 - 1) \); but the pattern of nice cumulant factorizations stops there. There's a much simpler...