It is noteworthy that a Fourier series may converge to a sum that is not differentiable, or even continuous, in spite of the fact that each term in the series (4) is continuous, and even differentiable infinitely many times. The example below is an illustration of this, as is Example 2 in Section 10.2.

**Example 1**

Let 

\[ f(x) = \begin{cases} 
0, & -L < x < 0, \\
L, & 0 < x < L 
\end{cases} \]  

(5)

and let \( f \) be defined outside this interval so that \( f(x + 2L) = f(x) \) for all \( x \). We will temporarily leave open the definition of \( f \) at the points \( x = 0, \pm L \), except that its value must be finite. Find the Fourier series for this function and determine where it converges.

**FIGURE 10.3.2** Square wave.

The equation \( y = f(x) \) has the graph shown in Figure 10.3.2, extended to infinity in both directions. It can be thought of as representing a square wave. The interval \([-L, L]\) can be partitioned to give the two open subintervals \((-L, 0)\) and \((0, L)\). In \((0, L)\), \( f(x) = L \) and \( f'(x) = 0 \). Clearly, both \( f \) and \( f' \) are continuous and furthermore have limits as \( x \to 0 \) from the right and as \( x \to L \) from the left. The situation in \((-L, 0)\) is similar. Consequently, both \( f \) and \( f' \) are piecewise continuous on \([-L, L]\), so \( f \) satisfies the conditions of Theorem 10.3.1. If the coefficients \( a_m \) and \( b_m \) are computed from Eqs. (2) and (3) the convergence of the resulting Fourier series to \( f(x) \) is assured at all points where \( f \) is continuous. Note that the values of \( a_m \) and \( b_m \) are the same regardless of the definition of \( f \) at its points of discontinuity. This is true because the value of an integral is unaffected by changing the value of the integrand at a finite number of points. From Eq. (2)

\[ a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx = \int_{0}^{L} dx = L; \]

\[ a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} \, dx = \int_{0}^{L} \cos \frac{m\pi x}{L} \, dx = 0, \quad m \neq 0. \]

Similarly, from Eq. (3)

\[ b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} \, dx = \int_{0}^{L} \sin \frac{m\pi x}{L} \, dx = \frac{L}{m\pi} (1 - \cos m\pi) \]

\[ = \begin{cases} 0, & m \text{ even;} \\
2L/m\pi, & m \text{ odd.} \end{cases} \]