way to derive the mean: We have $G_m + A_1 + \ldots + A_l = z(A_1 + \ldots + A_l) + 1$, hence when $z = 1$ we have $G'_m = A_1 + \ldots + A_l$. Since $G_m = 1$ when $z = 1$, an easy induction shows that $A_{lk} = 4k$.

8.57 We have $A:A \geq 2^l + 1$ and $B:B < 2^l + 2^l + 3$ and $B:A \geq 2^{l+2}$, hence $B:B = B:A \geq A:A \geq A:B$ is possible only if $A:B > 2^{l-3}$. This means that $\tau_2 = \tau_3$, $\tau_1 = \tau_4$, $\tau_2 = \tau_5$, $\tau_3 = \tau_6$. But then $A:A \approx 2^l + 2^{l+4} + \ldots$, $A:B \approx 2^l + 2^{l+6} + 2^{l+8} + \ldots$, and $B:B \approx 2^{l-1} + 2^{l+4} + \ldots$; hence $B:B = B:A$ is less than $A:A = A:B$ after all. (Sharper results have been obtained by Guibas and Odlyzko [138], who show that Bill’s chances are always maximized with one of the two patterns $H\tau_1 \ldots \tau_l$ or $T\tau_l \ldots$.)

8.58 According to (8.82), we want $B:B > B:A > A:A > A:B$. One solution is $A = THHH$, $B = HHH$.

8.59 (a) Two cases arise depending on whether $h_{k-1} \neq h_n$ or $h_{k-1} = h_n$:

$$G(w, z) = \frac{m - 1}{m} \left( \frac{m - 2 + w}{m} \right)^{z-1} w \left( \frac{m - 1 + z}{m} \right)^{n-k} \frac{1}{z}$$

(b) We can either argue algebraically, taking partial derivatives of $G(w, z)$ with respect to $w$ and $z$ and setting $w = z = 1$; or we can argue combinatorially: Whatever the values of $h_1, \ldots, h_{n-1}$, the expected value of $P(h_1, \ldots, h_{n-1}, h_n; n)$ is the same (averaged over $h_n$), because the hash sequence $(h_1, \ldots, h_{n-1})$ determines a sequence of list sizes $(n_1, n_2, \ldots, n)$ such that the stated expected value is $((n_1+1) + (n_2+1) + \ldots + (n_{n-1}+1))/n = (n - 1 + m)/m$. Therefore the random variable $E[P(h_1, \ldots, h_{n-1}; n)]$ is independent of $(h_1, \ldots, h_{n-1})$, hence independent of $P(h_1, \ldots, h_{n-1}; k)$.

8.60 If $1 \leq k \leq n$, the previous exercise shows that the coefficient of $S_k S_1$ in the variance of the average is zero. Therefore we need only consider the coefficient of $S_k^2$, which is

$$\sum_{1 \leq h_1, \ldots, h_n \leq m} \frac{P(h_1, \ldots, h_n; k)^2}{m^n} - \left( \sum_{1 \leq h_1, \ldots, h_n \leq m} \frac{P(h_1, \ldots, h_n; k)}{m^n} \right)^2,$$

defining the variance of $((m - 1 + z)/m)^{k-1} z$; and this is $(k - 1)(m - 1)/m^2$ as in exercise 30.

8.61 The pgf $D_0(z)$ satisfies the recurrence

$$D_0(z) = z;$$
$$D_n(z) = z^2 D_{n-1}(z) + 2(1 - z^3) D'_{n-1}(z)/(n + 1), \quad \text{for } n > 0.$$