10.3 The Fourier Convergence Theorem

- 8. \( f(x) = \begin{cases} x + 1, & -1 \leq x < 0, \\ 1 - x, & 0 \leq x < 1; \end{cases} \) \( f(x + 2) = f(x) \) (see Section 10.2, Problem 16)

- 9. \( f(x) = x, \) \( -1 \leq x < 1; \) \( f(x + 2) = f(x) \) (see Section 10.2, Problem 20)

- 10. \( f(x) = \begin{cases} x + 2, & -2 \leq x < 0, \\ 2 - 2x, & 0 \leq x < 2; \end{cases} \) \( f(x + 4) = f(x) \) (see Section 10.2, Problem 22)

- 11. \( f(x) = \begin{cases} 0, & -1 \leq x < 0, \\ x^2, & 0 \leq x < 1; \end{cases} \) \( f(x + 2) = f(x) \) (see Problem 6)

- 12. \( f(x) = x - x^3, \) \( -1 \leq x < 1; \) \( f(x + 2) = f(x) \)

**Periodic Forcing Terms.** In this chapter we are concerned mainly with the use of Fourier series to solve boundary value problems for certain partial differential equations. However, Fourier series are also useful in many other situations where periodic phenomena occur. Problems 13 through 16 indicate how they can be employed to solve initial value problems with periodic forcing terms.

13. Find the solution of the initial value problem

\[ y'' + \omega^2 y = \sin nt, \quad y(0) = 0, \quad y'(0) = 0, \]

where \( n \) is a positive integer and \( \omega^2 \neq n^2 \). What happens if \( \omega^2 = n^2 \)?

14. Find the formal solution of the initial value problem

\[ y'' + \omega^2 y = \sum_{n=1}^{\infty} b_n \sin nt, \quad y(0) = 0, \quad y'(0) = 0, \]

where \( \omega > 0 \) is not equal to a positive integer. How is the solution altered if \( \omega = m \), where \( m \) is a positive integer?

15. Find the formal solution of the initial value problem

\[ y'' + \omega^2 y = f(t), \quad y(0) = 0, \quad y'(0) = 0, \]

where \( f \) is periodic with period \( 2\pi \) and

\[ f(t) = \begin{cases} 1, & 0 < t < \pi; \\ 0, & t = 0, \pi, 2\pi; \\ -1, & \pi < t < 2\pi. \end{cases} \]

See Problem 1.

16. Find the formal solution of the initial value problem

\[ y'' + \omega^2 y = f(t), \quad y(0) = 1, \quad y'(0) = 0, \]

where \( f \) is periodic with period 2 and

\[ f(t) = \begin{cases} 1 - t, & 0 \leq t < 1; \\ -1 + t, & 1 \leq t < 2. \end{cases} \]

See Problem 8.

17. Assuming that

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \]

show formally that

\[ \frac{1}{L} \int_{-L}^{L} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \]