which is $2O(f(n) \sum_{k \geq 0} |f(k)|)$, so this case is proved. (b) But in this case if $a_n - b_n = \alpha^{-n}$, the convolution $(n + 1)\alpha^{-n}$ is not $O(\alpha^n)$.

9.25 $S_n/(\binom{3n}{n}) = \sum_{k=0}^{n} \frac{n^k}{(2n+1)^k}$. We may restrict the range of summation to $0 \leq k \leq (\log n)^2$, say. In this range $n^k = n^k(1 + \frac{k^2}{2n} + O(k^4/n^2))$ and $(2n+1)^k = (2n)^k(1 + \frac{k^2}{2n} + O(k^4/n^2))$, so the summand is

$$\frac{1}{2^k} \left(1 - \frac{3k^2}{4n} + O\left(\frac{k^4}{n^2}\right)\right)$$

Hence the sum over $k$ is $2^{-4/n} + O(1/n^2)$. Stirling’s approximation can now be applied to $(\binom{3n}{n}) = [3n!/((2n)!n!)]$, proving (g,2).

9.26 The minimum occurs at a term $B_{2m}/(2m)(2m - 1)n^{2m-1}$ where $2m \approx 2\pi n + \frac{1}{2}$, and this term is approximately equal to $1/(\pi e^{2\pi n} n^{1/2})$. The absolute error in $\ln n!$ is therefore too large to determine $n!$ exactly by rounding to an integer, when $n$ is greater than about $e^{2\pi n}$.

9.27 We may assume that $a \neq -1$. Let $f(x) = x^a$; the answer is

$$\sum_{k=1}^{n} k^a = C_a + \frac{n^{a+1}}{a+1} + \frac{n^a}{2} + \sum_{k=1}^{m} \frac{B_{2k}}{2k} \binom{a}{2k-1} n^{a-2k+1} + O(n^{a-2m+1}).$$

(The constant $C_a$ turns out to be $\zeta(-a)$, which is in fact defined by this formula when $a > -1$.)

9.28 Take $f(x) = x\ln x$ in Euler’s summation formula to get

$$A \cdot n^{\gamma/2 + n/2 + 1/12} e^{-n/4}(1 + O(n^{-2})),$$

where $A \approx 1.282427$ is “Glaisher’s constant!”

9.29 Let $f(x) = x^{-1} \ln x$. Then $f^{[2m]}(x) > 0$ for all large $x$, and we can write

$$\sum_{k=1}^{n} \frac{\ln k}{k} = \frac{(\ln n)^2}{2} + \ln S + \frac{\ln n}{2n} + \theta_n \frac{1 - \ln n}{12n^2}, \quad 0 < \theta_n < 1,$$

where $S \approx 0.929772$ is constant. Taking exponentials gives

$$S \sqrt{n \ln n} \left(1 + \frac{\ln n}{2n} + O\left(\frac{\log n}{n}\right)^2\right).$$

(In general if $f(x) = x^a \ln x$, Euler’s summation formula applies as in exercise 27, and the resulting constant is $\zeta(-a)$ if $a \neq -1$. Thus, the theory of the zeta function gives a closed form for Glaisher’s constant in the previous exercise. We have $\ln S = \gamma_1$ in the notation of answer 9.57.)