Hence \( S_n = \frac{1}{4} \pi n^{-1} - \frac{1}{4} n^{-2} - \frac{1}{24} n^{-3} + O(n^{-5}) \).

9.37 This is

\[
\sum_{k,q \geq 1} (n - qk) \left[ \frac{n}{(q + 1)} < k \leq \frac{n}{q} \right]
\]

\[= n^2 \sum_{q \geq 1} q \left( \left\lfloor \frac{n}{q} \right\rfloor + 1 \right) - \left( \left\lfloor \frac{n}{(q + 1)} \right\rfloor + 1 \right)\]

\[= n^2 \sum_{q \geq 1} \left( \left\lfloor \frac{n}{q} \right\rfloor + 1 \right).
\]

The remaining sum is like (9.55) but without the factor \( u(q) \). The same method works here as it did there, but we get \( \zeta(2) \) in place of \( 1/\zeta(2) \), so the answer comes to \( (1 - \frac{\pi^2}{12}) n^2 + O(n \log n) \).

9.38 Replace \( k \) by \( n - k \) and let \( \omega(n) = (n - k)^{n-k} \binom{n}{k} \). Then

\[\ln \omega(n) = n \ln n - \ln k! - k + O(k n^{-1})\]

and we can use tail-exchange with \( b_k(n) = n^n e^{-k/\log n} \), \( c_k(n) = k b_k(n)/n \), \( D_n = \{ k : 0 \leq k \leq \ln n \} \), to get

\[\sum_{k=0}^{n} \omega(n) = n^n e^{1/e} (1 + O(n^{-1})).\]

9.39 Tail-exchange with \( b_k(n) = (\ln n - k/n - \ln k!)^k/\ln n \), \( c_k(n) = n^{-3} (\ln n)^{k+1}/k! \), \( D_n = \{ k : 0 \leq k \leq \ln n \} \). When \( k \approx 10 \ln n \), we have \( k! \approx k^k (10/e)^k (\ln n)^k \), so the kth term is \( O(n^{-10 \ln(10/e) \ln n}) \). The answer is \( n n^n - n - 1/2 \ln n [1 + \ln n] / n + O(n^{-2} \log n^3) \).

9.40 Combining terms two by two, we find that \( H_{2k}^m - H_{2k}^{m-1} \)

plus terms whose sum over all \( k \geq 1 \) is \( O(1) \). Suppose \( n \) is even. Euler's summation formula implies that

\[\sum_{k=1}^{\lfloor n/2 \rfloor} H_{2k}^m - H_{2k}^{m-1} = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(\ln 2 e^\gamma)^{m-1} + O(1/k)}{k} + O(1) = \frac{(\ln e^\gamma n)^m}{m} + O(1)\]

hence the sum is \( \frac{1}{2} H_n^m + O(1) \). In general the answer is \( \frac{1}{2} (-1)^n H_n^m + O(1) \).

9.41 Let \( \alpha = \phi/\phi = -\phi^{-2} \). We have

\[\sum_{k=1}^{n} \ln F_k = \sum_{k=1}^{n} \ln \phi^k - \ln \sqrt{5} + \ln (1 - \alpha^k)\]

\[= \frac{n(n+1)}{2} \ln \phi - \frac{n}{2} \ln 5 + \sum_{k \geq 1} \ln (1 - \alpha^k) - \sum_{k > n} \ln (1 - \alpha^k)\]

The latter sum is \( \sum_{k > n} O(\alpha^k) = O(\alpha^n) \). Hence the answer is

\[\phi^{n(n+1)/2} 5^{-n/2} C + O(\phi^{n(n+1)/2} 5^{-n/2})\]

where

\[C = (1 - \alpha)(1 - \alpha^2)(1 - \alpha^3) \cdots \approx 1.226742.\]