and the Fourier series for $f$ is of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$ 

Thus the Fourier series for any odd function consists only of the odd trigonometric functions $\sin(n\pi x / L)$; such a series is called a **Fourier sine series**. Again observe that only half of the coefficients need to be calculated by integration, since each $a_n$, for $n = 0, 1, 2, \ldots$, is zero for any odd function.

**Example 1**

Let $f(x) = x, -L < x < L$, and let $f(-L) = f(L) = 0$. Let $f$ be defined elsewhere so that it is periodic of period $2L$ (see Figure 10.4.2). The function defined in this manner is known as a sawtooth wave. Find the Fourier series for this function.

Since $f$ is an odd function, its Fourier coefficients are, according to Eq. (8),

$$a_n = 0, \quad n = 0, 1, 2, \ldots;$$

$$b_n = \frac{2}{L} \int_{0}^{L} x \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{2}{L} \left[ \frac{L}{n\pi} \right]^2 \left\{ \sin \frac{n\pi x}{L} - \frac{n\pi x}{L} \cos \frac{n\pi x}{L} \right\} \bigg|_{0}^{L}$$

$$= \frac{2L}{n\pi} (-1)^{n+1}, \quad n = 1, 2, \ldots.$$

Hence the Fourier series for $f$, the sawtooth wave, is

$$f(x) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L}. \quad (9)$$

Observe that the periodic function $f$ is discontinuous at the points $\pm L, \pm 3L, \ldots$, as shown in Figure 10.4.2. At these points the series (9) converges to the mean value of the left and right limits, namely, zero. The partial sum of the series (9) for $n = 9$ is shown in Figure 10.4.3. The Gibbs phenomenon (mentioned in Section 10.3) again occurs near the points of discontinuity.