9.42 The hint follows since \( \binom{n}{k} / \binom{n}{k-1} = \frac{k}{n-k+1} \leq \frac{\alpha}{\alpha_{n+1}} < \frac{\alpha}{1-\alpha} \). Let \( m = \lfloor \alpha n \rfloor = \alpha n - \varepsilon \). Then
\[
\binom{n}{m} < \sum_{k \leq m} \binom{n}{k} < \binom{n}{m} \left(1 + \frac{\alpha}{1-\alpha} + \left(\frac{\alpha}{1-\alpha}\right)^2 + \cdots\right) = \frac{\binom{n}{m}}{1 - \frac{\alpha}{1-\alpha}}.
\]
So \( \sum_{k \leq n} \binom{n}{k} = (\binom{n}{m} + 1) \) and it remains to estimate \( \binom{n}{m} \). By Stirling’s approximation we have
\[
\ln(\binom{n}{m}) = -\frac{1}{2} \ln n - \left(\alpha n - \varepsilon\right) \ln(\alpha - \varepsilon / n) - \left((1-\alpha)n + \varepsilon\right) \ln(1 - \alpha + \varepsilon / n) + O(1).
\]
9.43 The denominator has factors of the form \( z - \omega \), where \( \omega \) is a complex root of unity. Only the factor \( z - 1 \) occurs with multiplicity 5. Therefore by (7.31), only one of the roots has a coefficient \( \Omega(n^4) \), and the coefficient is \( \varepsilon = 5/(5 \cdot 1 \cdot 5 \cdot 10 \cdot 25 \cdot 50) = 1/1500000 \).

9.44 Stirling’s approximation says that \( \ln(x^{-\alpha}!/(x-\alpha)!)/x! \) has an asymptotic series
\[
-\alpha - (x + \frac{1}{2} - \alpha) \ln(1 - \alpha/x) - \frac{B_2}{2 \cdot 1} \left(x^{1 - (x - \alpha)} - 1\right) - \frac{B_4}{4 \cdot 3} \left(x^3 - (x - \alpha)^3\right) + \ldots,
\]
in which each coefficient of \( x^{-k} \) is a polynomial in \( \alpha \). Hence \( x^{\alpha}!/(x-\alpha)! = c_0(\alpha) + c_1(\alpha)x^{-1} + \cdots + c_n(\alpha)x^{-n} + O(x^{-n-1}) \) as \( x \to \infty \), where \( c_n(\alpha) \) is a polynomial in \( \alpha \). We know that \( c_n(\alpha) = \left[\alpha^{-n}\right](n-2) \) whenever \( \alpha \) is an integer, and \( \left[\alpha^{-n}\right] \) is a polynomial in \( \alpha \) of degree \( 2n \); hence \( c_n(\alpha) = \left[\alpha^{-n}\right](-1)^n \) for all real \( \alpha \). In other words, the asymptotic formulas
\[
\begin{align*}
\alpha^x &= \sum_{k=0}^{\infty} \frac{\alpha^k}{\alpha - k}\left(-1\right)^{k+1}x^{a-k} + O(x^{a-n-1}), \\
x^\alpha &= \sum_{k=0}^{\infty} \frac{\alpha^k}{\alpha - k}x^{ak} + O(x^{a-n-1})
\end{align*}
\]
generalize equations (6.13) and (6.11), which hold in the all-integer case.

9.45 Let the partial quotients of \( 1/x \) be \( (a_1, a_2, \ldots) \), and let \( \alpha_m \) be the continued fraction \( 1/(a_m + a_{m+1}) \) for \( m \geq 1 \). Then \( D(\alpha, n) = D(\alpha_1, n) < D(\alpha_2, [\alpha_1, n]) + a_1 + 3 < D(\alpha_3, [\alpha_2, [\alpha_1, n]]) + a_1 + a_2 + 6 < \cdots < D(\alpha_{m+1}, [\alpha_m, \ldots, [\alpha_1, n] \ldots]) + a_1 + \cdots + a_m + 3m < \alpha_1 \ldots \alpha_m + a_1 + \cdots + a_m + 3m, \)