9.42 The hint follows since \( \binom{n}{k}/\binom{n}{k+1} = \frac{k}{n-k+1} \leq \frac{\alpha n}{n-\alpha n+1} < \frac{\alpha}{1-\alpha} \). Let \( m = [\alpha n] = \alpha n - \epsilon \). Then

\[
\binom{n}{m} < \sum_{k \leq m} \binom{n}{k} < \binom{n}{m} \left(1 + \frac{\alpha}{1-\alpha} + \left(\frac{\alpha}{1-\alpha}\right)^2 + \cdots\right) = \binom{n}{m} \frac{1-\alpha}{1-2\alpha}.
\]

So \( \sum_{k \leq \alpha n} \binom{n}{k} = \binom{n}{m} \Omega(1) \), and it remains to estimate \( \binom{n}{m} \). By Stirling’s approximation we have

\[
\ln \left(\binom{n}{m}\right) = -\frac{1}{2} \ln n - \left(\alpha n - \epsilon\right) \ln (\alpha n - \epsilon) - \left(1 - \alpha n + \epsilon\right) \ln (1 - \alpha n + \epsilon) + O(1) \approx \frac{1}{2} \ln n - \alpha n \ln \alpha - (1 - \alpha) n \ln (1 - \alpha) + O(1).
\]

9.43 The denominator has factors of the form \( z - w \), where \( w \) is a complex root of unity. Only the factor \( z - 1 \) occurs with multiplicity 5. Therefore by (7.31), only one of the roots has a coefficient \( \Omega(n^4) \), and the coefficient is \( c = 5/(5! \cdot 5 \cdot 10 \cdot 25 \cdot 50) = 1/1500000 \).

9.44 Stirling’s approximation says that \( \ln \left(\frac{x^\alpha x!}{(x-\alpha)!}\right) \) has an asymptotic series

\[
-\alpha - (x + \frac{1}{2} - \alpha) \ln (1 - \alpha/x) - \frac{B_2}{2!} \left(x^1 - (x-\alpha)^{-1}\right) - \frac{B_4}{4!} \left(x^3 - (x - \alpha)^3\right) + \cdots,
\]

in which each coefficient of \( x^{-k} \) is a polynomial in \( \alpha \). Hence \( x^\alpha x!/(x-\alpha)! = c_0(\alpha) + c_1(\alpha)x + \cdots + c_n(\alpha)x^n + O(x^{-1}) \) as \( x \to \infty \), where \( c_n(\alpha) \) is a polynomial in \( \alpha \). We know that \( c_n(\alpha) = \left[ \frac{\alpha}{n} \right] (-1)^n \) whenever \( \alpha \) is an integer, and \( \left[ \frac{\alpha}{n} \right] \) is a polynomial in \( \alpha \) of degree \( 2n \); hence \( c_n(\alpha) = \left[ \frac{\alpha}{n} \right] (-1)^n \) for all real \( \alpha \). In other words, the asymptotic formulas

\[
\begin{align*}
 x^\alpha &= \sum_{k=0}^{n} \left[ \frac{\alpha}{\alpha - k} \right] (-1)^k x^{\alpha - k} + O(x^{\alpha - 1}), \\
 x^{\alpha^n} &= \sum_{k=0}^{n} \left[ \frac{\alpha}{\alpha - k} \right] x^{\alpha k} + O(x^{\alpha - n - 1})
\end{align*}
\]

generalize equations (6.13) and (6.11), which hold in the all-integer case.

9.45 Let the partial quotients of \( 1x \) be \( \langle a_1, a_2, \ldots \rangle \), and let \( \alpha_m \) be the continued fraction \( 1/(a_m + \alpha_m+1) \) for \( m \geq 1 \). Then \( D(\alpha, n) = D(\alpha_1, n) < \alpha_1 + 3 < \alpha_3 + [\alpha_2 [\alpha_1 n]] + a_1 + a_2 + 6 < \cdots < \alpha_m + a_1 + a_2 + a_3 + 3m < \alpha_1 \cdots \alpha_m n + a_1 + \cdots + a_m + 3m \).