for all \( m \). Divide by \( n \) and let \( n \to \infty \); the limit is less than \( \alpha_1 \ldots \alpha_m \) for all \( m \). Finally we have
\[
\alpha_1 \ldots \alpha_m = \frac{1}{K(\alpha_1, \ldots, \alpha_m, \alpha_m + \alpha_m)} < \frac{1}{f_{m+1}}
\]

9.46 For convenience we write just \( m \) instead of \( m(n) \). By Stirling’s approximation, the maximum value of \( k^n/k! \) occurs when \( k \approx m \approx n/\ln n \), so we replace \( k \) by \( m + k \) and find that
\[
\ln \frac{(m+k)^n}{(m+k)!} = n \ln m - m \ln n + m - \frac{\ln 2 \pi m}{2}
\]
\[
\frac{1}{2} \frac{(m+n)k^2}{2m^2} + O(k^3 m^{3/2} \log n)
\]
Actually we want to replace \( k \) by \( \lfloor m \rfloor + k \); this adds a further \( O(k^3 \log \log n) \).

The requested formula follows, with relative error \( 0 (\log \log n/\log n) \).

9.47 Let \( \log_m n = l + \theta \), where \( 0 \leq \theta < 1 \). The floor sum is \( l(n+1) + 1 - (m^l - 1)/(m - 1) \); the ceiling sum is \( (l+1)n - (m^{l+1} - 1)/(m - 1) \); the exact sum is \( (l+\theta)n - n/\ln m + O(\log n) \). Ignoring terms that are \( o(n) \), the difference between ceiling and exact is \( (1 - f(\theta)) n \), and the difference between exact and floor is \( f(\theta) n \), where
\[
f(\theta) = \frac{m^{1-\theta}}{m - 1} + \theta - \frac{1}{\ln m}
\]
This function has maximum value \( f(0) = f(1) = m/(m - 1) \), \( 1/\ln m \), and its minimum value is \( \ln \ln m/\ln m + 1 - (\ln(m - 1))/\ln m \). The ceiling value is closer when \( n \) is nearly a power of \( m \), but the floor value is closer when \( \theta \) lies somewhere between 0 and 1.

9.48 Let \( d_k = a_k + b_k \), where \( a_k \) counts digits to the left of the decimal point. Then \( a_k = 1 + \lfloor \log_{10} k \rfloor = \log \log k + O(1) \), where ‘log’ denotes \( \log_{10} \).
To estimate \( b_k \), let us look at the number of decimal places necessary to distinguish \( y \) from nearby numbers \( y - \varepsilon \) and \( y + \varepsilon' \). Let \( \delta = 10 \delta \) be the