A ANSWERS TO EXERCISES 577

9.62 See Canfield [43]; see also David and Barton [60, Chapter 16] for asymptotics of Stirling numbers of both kinds.

9.63 Let $c = \phi^2 - \phi$. The estimate $cn^{\phi-1} + o(n^{\phi-1})$ was proved by Fine [120]. Ilan Vardi observes that the sharper estimate stated can be deduced from the fact that the error term $e(n) = f(n) - cn^{\phi-1}$ satisfies the approximate recurrence $c^n n^{2-\phi} e(n) \approx \sum_{\kappa} e(\kappa) [1 \leq \kappa < cn^{\phi-1}]$. The function

$$\frac{n^{\phi-1} u(\ln \ln n / \ln \phi)}{\ln n}$$

satisfies this recurrence asymptotically, if $u(x + 1) = -u(x)$. (Vardi conjectures that

$$f(n) = n^{\phi-1} \left( c + u\left( \frac{\ln \ln n}{\ln \phi} \right) \left( \frac{\ln n}{\ln \phi} \right)^{-1} + O\left( (\log n)^{-2} \right) \right)$$

for some such function $u$.) Calculations for small $n$ show that $f(n)$ equals the nearest integer to $cn^{\phi-1}$ for $1 \leq n \leq 400$ except in one case: $f(273) = 39 > c \cdot 273^{\phi-1} \approx 38.4997$. But the small errors are eventually magnified, because of results like those in exercise 2.36. For example, $e(201636503) \approx 5573; e(919986484788) \approx -1959.07.$

9.64 (From this identity for $B_2(x)$ we can easily derive the identity of exercise 58 by induction on $m$.) If $0 < x < 1$, the integral $\int_0^{1/2} \sin N \pi t dt / \sin \pi t$ can be expressed as a sum of $N$ integrals that are each $0 (N^{-2})$, so it is $O (N^{-1})$; the constant implied by this $0$ may depend on $x$. Integrating the identity

$$\sum_{n=1}^{N} \cos 2n \pi t = \Re \left( e^{2 \pi i t} (e^{2N \pi it} - 1) / (e^{2 \pi it} - 1) \right) = -\frac{1}{2} + \frac{1}{2} \sin (2N+1) \pi t / \sin \pi t$$

and letting $N \to \infty$ gives $\sum_{n \geq 1} (\sin 2n \pi x) / n = \frac{\pi}{2} - \pi x$, a relation that Euler knew ([85']) and ([88, part 2, §92]). Integrating again yields the desired formula. (This solution was suggested by E. M. E. Wermuth; Euler’s original derivation did not meet modern standards of rigor.)

9.65 The expected number of distinct elements in the sequence $1, f(1), f(f(1)), \ldots$, when $f$ is a random mapping of $\{1, 2, \ldots, n\}$ into itself, is the function $Q(n)$ of exercise 56, whose value is $\frac{1}{2} \sqrt{2\pi n} + O(1)$; this might account somehow for the factor $\sqrt{2\pi n}$.

9.66 It is known that $\ln \gamma_n \sim \frac{1}{2} n^2 \ln \frac{4}{3}$; the constant $e^{-\gamma/6}$ has been verified empirically to eight significant digits.

9.67 This would fail if, for example, $e^{n-\gamma} = m + \frac{1}{2} + \varepsilon / m$ for some integer $m$ and some $0 < \varepsilon < \frac{1}{8}$; but no counterexamples are known.