10.6 Other Heat Conduction Problems

In Section 10.5 we considered the problem consisting of the heat conduction equation
\[ \alpha^2 u_{xx} = u_t, \quad 0 < x < L, \quad t > 0, \quad (1) \]
the boundary conditions
\[ u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0, \quad (2) \]
and the initial condition
\[ u(x, 0) = f(x), \quad 0 \leq x \leq L. \quad (3) \]
We found the solution to be
\[ u(x, t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha^2 t / L^2} \sin \frac{n\pi x}{L}, \quad (4) \]
where the coefficients \( c_n \) are the same as in the series
\[ f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}. \quad (5) \]
The series in Eq. (5) is just the Fourier sine series for \( f \); according to Section 10.4 its coefficients are given by
\[ c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx. \quad (6) \]
Hence the solution of the heat conduction problem, Eqs. (1) to (3), is given by the series in Eq. (4) with the coefficients computed from Eq. (6).

We emphasize that at this stage the solution (4) must be regarded as a formal solution; that is, we obtained it without rigorous justification of the limiting processes involved.