Such a justification is beyond the scope of this book. However, once the series (4) has been obtained, it is possible to show that in $0 < x < L$, $t > 0$ it converges to a continuous function, that the derivatives $u_{xx}$ and $u_t$ can be computed by differentiating the term by term, and that the heat conduction equation (1) is indeed satisfied. The argument rests heavily on the fact that each term of the series (4) contains a negative exponential factor, and this results in relatively rapid convergence of the series. A further argument establishes that the function $u$ given by Eq. (4) also satisfies the boundary and initial conditions; this completes the justification of the formal solution.

It is interesting to note that although $f$ satisfies the conditions of the Fourier convergence theorem (Theorem 10.3.1), it may have points of discontinuity. In this case the initial temperature distribution $u(x, 0) = f(x)$ is discontinuous at one or more points. Nevertheless, the solution $u(x, t)$ is continuous for arbitrarily small values of $t > 0$. This illustrates the fact that heat conduction is a diffusive process that instantly smooths out any discontinuities that may be present in the initial temperature distribution. Finally, since $f$ is bounded, it follows from Eq. (6) that the coefficients $c_n$ are also bounded. Consequently, the presence of the negative exponential factor in each term of the series (4) guarantees that

$$\lim_{t \to \infty} u(x, t) = 0$$  \hspace{1cm} (7)

for all $x$ regardless of the initial condition. This is in accord with the result expected from physical intuition.

We now consider two other problems of one-dimensional heat conduction that can be handled by the method developed in Section 10.5.

**Nonhomogeneous Boundary Conditions.** Suppose now that one end of the bar is held at a constant temperature $T_1$ and the other is maintained at a constant temperature $T_2$. Then the boundary conditions are

$$u(0, t) = T_1, \quad u(L, t) = T_2, \quad t > 0.$$  \hspace{1cm} (8)

The differential equation (1) and the initial condition (3) remain unchanged.

This problem is only slightly more difficult, because of the nonhomogeneous boundary conditions, than the one in Section 10.5. We can solve it by reducing it to a problem having homogeneous boundary conditions, which can then be solved as in Section 10.5. The technique for doing this is suggested by the following physical argument.

After a long time, that is, as $t \to \infty$, we anticipate that a steady temperature distribution $v(x)$ will be reached, which is independent of the time $t$ and the initial conditions. Since $v(x)$ must satisfy the equation of heat conduction (1), we have

$$v''(x) = 0, \quad 0 < x < L.$$  \hspace{1cm} (9)

Hence the steady-state temperature distribution is a linear function of $x$. Further, $v(x)$ must satisfy the boundary conditions

$$v(0) = T_1, \quad v(L) = T_2,$$  \hspace{1cm} (10)

which are valid even as $t \to \infty$. The solution of Eq. (9) satisfying Eqs. (10) is

$$v(x) = (T_2 - T_1) \frac{x}{L} + T_1.$$  \hspace{1cm} (11)