and substitute for \( u \) in Eq. (1), then it follows as in Section 10.5 that

\[
\frac{X''}{X} = \frac{1}{\alpha^2} \frac{T'}{T} = -\lambda, \tag{26}
\]

where \( \lambda \) is a constant. Thus we obtain again the two ordinary differential equations

\[
X'' + \lambda X = 0, \tag{27}
\]

\[
T' + \alpha^2 \lambda T = 0. \tag{28}
\]

For any value of \( \lambda \) a product of solutions of Eqs. (27) and (28) is a solution of the partial differential equation (1). However, we are interested only in those solutions that also satisfy the boundary conditions (24).

If we substitute for \( u(x, t) \) from Eq. (25) in the boundary condition at \( x = 0 \), we obtain

\[
X'(0)T(t) = 0. \tag{29}
\]

We cannot permit \( T(t) \) to be zero for all \( t \), since then \( u(x, t) \) would also be zero for all \( t \). Hence we must have

\[
X'(0) = 0. \tag{29}
\]

Proceeding in the same way with the boundary condition at \( x = L \), we find that

\[
X'(L) = 0. \tag{30}
\]

Thus we wish to solve Eq. (27) subject to the boundary conditions (29) and (30). It is possible to show that nontrivial solutions of this problem can exist only if \( \lambda \) is real. One way to show this is indicated in Problem 18; alternatively, one can appeal to a more general theory to be discussed later in Section 11.2. We will assume that \( \lambda \) is real and consider in turn the three cases \( \lambda < 0, \lambda = 0, \) and \( \lambda > 0. \)

If \( \lambda < 0 \), it is convenient to let \( \lambda = -\mu^2 \), where \( \mu \) is real and positive. Then Eq. (27) becomes

\[
X'' - \mu^2 X = 0 \tag{31}
\]

and its general solution is

\[
X(x) = k_1 \sinh \mu x + k_2 \cosh \mu x. \tag{31}
\]

In this case the boundary conditions can be satisfied only by choosing \( k_1 = k_2 = 0 \). Since this is unacceptable, it follows that \( \lambda \) cannot be negative; in other words, the problem (27), (29), (30) has no negative eigenvalues.

If \( \lambda = 0 \), then Eq. (27) is \( X'' = 0 \), and its general solution is

\[
X(x) = k_1 x + k_2. \tag{32}
\]

The boundary conditions (29) and (30) require that \( k_1 = 0 \) but do not determine \( k_2. \) Thus \( \lambda = 0 \) is an eigenvalue, corresponding to the eigenfunction \( X(x) = 1. \) For \( \lambda = 0 \) it follows from Eq. (28) that \( T(t) \) is also a constant, which can be combined with \( k_2. \) Hence, for \( \lambda = 0 \), we obtain the constant solution \( u(x, t) = k_2. \)

Finally, if \( \lambda > 0 \), let \( \lambda = \mu^2 \), where \( \mu \) is real and positive. Then Eq. (27) becomes

\[
X'' + \mu^2 X = 0, \tag{32}
\]

and its general solution is

\[
X(x) = k_1 \sin \mu x + k_2 \cos \mu x. \tag{33}
\]

The boundary conditions (29) require that \( k_1 = 0, \) and the boundary condition (30) requires that \( \mu = n\pi/L \) for \( n = 1, 2, 3, \ldots, \) but leaves \( k_2 \) arbitrary. Thus the problem (27), (29), (30) has an infinite sequence of positive eigenvalues \( \lambda = n^2 \pi^2/L^2 \) with the corresponding eigenfunctions \( X(x) = \cos(n\pi x/L) \). For these values of \( \lambda \) the solutions \( T(t) \) of Eq. (28) are proportional to \( \exp(-n^2 \pi^2 \alpha^2 t/L^2) \).