Combining all these results, we have the following fundamental solutions for the problem (1), (3), and (24):

\[
\begin{align*}
  u_0(x, t) &= 1, \\
  u_n(x, t) &= e^{-n^2\pi^2a^2t/L^2} \cos \frac{n\pi x}{L}, \quad n = 1, 2, \ldots ,
\end{align*}
\]

where arbitrary constants of proportionality have been dropped. Each of these functions satisfies the differential equation (1) and the boundary conditions (24). Because both the differential equation and boundary conditions are linear and homogeneous, any finite linear combination of the fundamental solutions satisfies them. We will assume that this is true for convergent infinite linear combinations of fundamental solutions as well. Thus, to satisfy the initial condition (3), we assume that \(u(x, t)\) has the form

\[
  u(x, t) = \frac{c_0}{2} u_0(x, t) + \sum_{n=1}^{\infty} c_n u_n(x, t)
\]

\[
= \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2a^2t/L^2} \cos \frac{n\pi x}{L}.
\]

The coefficients \(c_n\) are determined by the requirement that

\[
u(x, 0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{L} = f(x).
\]

Thus the unknown coefficients in Eq. (35) must be the coefficients in the Fourier cosine series of period \(2L\) for \(f\). Hence

\[
c_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx, \quad n = 0, 1, 2, \ldots .
\]

With this choice of the coefficients \(c_0, c_1, c_2, \ldots \), the series (35) provides the solution to the heat conduction problem for a rod with insulated ends, Eqs. (1), (3), and (24).

It is worth observing that the solution (35) can also be thought of as the sum of a steady-state temperature distribution (given by the constant \(c_0/2\)), which is independent of time \(t\), and a transient distribution (given by the rest of the infinite series) that vanishes in the limit as \(t\) approaches infinity. That the steady-state is a constant is consistent with the expectation that the process of heat conduction will gradually smooth out the temperature distribution in the bar as long as no heat is allowed to escape to the outside. The physical interpretation of the term

\[
c_0 = \frac{1}{L} \int_0^L f(x) \, dx
\]

is that it is the mean value of the original temperature distribution.