10.6 Other Heat Conduction Problems

11. Consider a rod of length 30 cm for which \( \alpha^2 = 1 \). Suppose the initial temperature distribution is given by \( u(x, 0) = x(60 - x)/30 \) and that the boundary conditions are \( u(0, t) = 30 \) and \( u(30, t) = 0 \).

(a) Find the temperature in the rod as a function of position and time.

(b) Plot \( u \) versus \( x \) for several values of \( t \). Also plot \( u \) versus \( t \) for several values of \( x \).

(c) Plot \( u \) versus \( x \) for several values of \( t \). Also plot \( u \) versus \( t \) for several values of \( x \).

(d) What limiting value does the temperature at the center of the rod approach after a long time? How much time must elapse before the center of the rod cools to within 1 degree of its limiting value?

12. Consider a uniform rod of length \( L \) with an initial temperature given by \( u(x, 0) = \sin(\pi x/L), \) \( 0 \leq x \leq L \). Assume that both ends of the bar are insulated.

(a) Find the temperature \( u(x, t) \).

(b) What is the steady-state temperature as \( t \to \infty \)?

(c) Let \( \alpha^2 = 1 \) and \( L = 40 \). Plot \( u \) versus \( x \) for several values of \( t \). Also plot \( u \) versus \( t \) for several values of \( x \).

(d) Describe briefly how the temperature in the rod changes as time progresses.

13. Consider a bar of length 40 cm whose initial temperature is given by \( u(x, 0) = x(60 - x)/30 \). Suppose that \( \alpha^2 = 1/4 \text{ cm}^2/\text{sec} \) and that both ends of the bar are insulated.

(a) Find the temperature \( u(x, t) \).

(b) Plot \( u \) versus \( x \) for several values of \( t \). Also plot \( u \) versus \( t \) for several values of \( x \).

(c) Determine the steady-state temperature in the bar.

(d) Determine how much time must elapse before the temperature at \( x = 40 \) comes within 1 degree of its steady-state value.

14. Consider a bar 30 cm long that is made of a material for which \( \alpha^2 = 1 \) and whose ends are insulated. Suppose that the initial temperature is zero except for the interval \( 5 < x < 10 \), where the initial temperature is 25°C.

(a) Find the temperature \( u(x, t) \).

(b) Plot \( u \) versus \( x \) for several values of \( t \). Also plot \( u \) versus \( t \) for several values of \( x \).

(c) Plot \( u(4, t) \) and \( u(11, t) \) versus \( t \). Observe that the points \( x = 4 \) and \( x = 11 \) are symmetrically located with respect to the initial temperature pulse, yet their temperature plots are significantly different. Explain physically why this is so.

15. Consider a uniform bar of length \( L \) having an initial temperature distribution given by \( f(x), 0 \leq x \leq L \). Assume that the temperature at the end \( x = 0 \) is held at 0°C, while the end \( x = L \) is insulated so that no heat passes through it.

(a) Show that the fundamental solutions of the partial differential equation and boundary conditions are

\[
\begin{align*}
    u_n(x, t) &= e^{-(2n - 1)^2 \pi^2 / 4L^2} \sin[(2n - 1)\pi x / 2L], \\
    &\quad n = 1, 2, 3, \ldots
\end{align*}
\]

(b) Find a formal series expansion for the temperature \( u(x, t) \),

\[
u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t),
\]

that also satisfies the initial condition \( u(x, 0) = f(x) \).

*Hint:* Even though the fundamental solutions involve only the odd sines, it is still possible